The Supersymmetric Method in Random Matrix Theory and Applications to QCD

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Abstract.
    The supersymmetric method is a powerful method for the nonperturbative evaluation of quenched averages in disordered systems. Among others, this method has been applied to the statistical theory of S-matrix fluctuations, the theory of universal conductance fluctuations and the microscopic spectral density of the QCD Dirac operator.

    We start this series of lectures with a general review of Random Matrix Theory and the statistical theory of spectra. An elementary introduction of the supersymmetric method in Random Matrix Theory is given in the second and third lecture. We will show that a Random Matrix Theory can be rewritten as an integral over a supermanifold. This integral will be worked out in detail for the Gaussian Unitary Ensemble that describes level correlations in systems with broken time reversal invariance. We especially emphasize the role of symmetries.

    As a second example of the application of the supersymemtric method we discuss the calculation of the microscopic spectral density of the QCD Dirac operator. This is the eigenvalue density near zero on the scale of the average level spacing which is known to be given by chiral Random Matrix Theory. Also in this case we use symmetry considerations to rewrite the generating function for the resolvent as an integral over a supermanifold.

    The main topic of the second last lecture is the recent developments on the relation between the supersymmetric partition function and integrable hierarchies (in our case the Toda lattice hierarchy). We will show that this relation is an efficient way to calculate superintegrals. Several examples that were given in previous lectures will be worked out by means of this new method. Finally, we will discuss the quenched QCD Dirac spectrum at nonzero chemical potential. Because of the nonhermiticity of the Dirac operator the usual supersymmetric method has not been successful in this case. However, we will show that the supersymmetric partition function can be evaluated by means of the replica limit of the Toda lattice equation.

CONTENTS

1. INTRODUCTION

Random Matrix Theories are ensembles of matrices with random numbers as matrix elements. They were first introduced in physics in 1955 by Wigner [1] to describe the spacing distribution of nuclear levels. The mathematical problem of Random Matrix Theory is to derive eigenvalue correlations in the limit of large matrices. This problem was solved by Dyson [2] and Mehta and Gaudin [3, 4] who introduced orthogonal polynomials to perform the matrix integrals. About the same time the Anderson model [5] was introduced to describe the localization of wave functions in one-dimensional systems which led to the theory of disordered systems. Random Matrix Theory, with its main applications to nuclear physics, and the theory of disordered systems, with its main applications to solid state physics, developed independently until the mid eighties when it was realized that Random Matrix Theory is the zero dimensional limit of a disordered system [7, 6, 8, 9]. This was made possible in particular by the introduction of a field theoretical formulation of the theory of the disordered systems [10], and by the introduction of the supersymmetric method which made it possible to perform nonperturbative calculations. After these developments a rapid growth of Random Matrix Theory followed in the late eighties after the discovery of universal conductance fluctuations [11] and applications of Random Matrix Theory to quantum gravity [12, 13]. Both applications greatly enhanced the intellectual breath of the subject.

Random Matrix Theory has many applications. We already mentioned the correlations of nuclear levels [14]. It was realized in 1985 [15] that the necessary ingredient for levels to be correlated according to Random Matrix Theory is not the complexity of the system but rather that the system is classically chaotic. This led to many applications in the field of Quantum Chaos. A second application of Random Matrix Theory in nuclear physics is the description of S-matrix fluctuations . This problem was solved by means of the supersymmetric method [9] and the same approach was later applied to the theory of universal conductance fluctuations [16].

A new class of Random Matrix Ensembles was introduced in the early nineties [17, 18] to describe the eigenvalue correlation of the Euclidean Dirac operator in QCD. This made it possible to derive exact analytical results for Dirac spectra that have been generated by means of lattice QCD Monte-Carlo simulations. Random Matrix Theory in this field made it possible to get a much better grip on finite size corrections.

In Mathematics, Random Matrix Theory was applied to the correlations of the non-trivial zeros of the Riemann ζ-function [19]. After the impressive numerical work by Odlyzko [20] this has been understood in great detail by Berry and Keating (see [21] for a review). More recently, Random Matrix Theory has been applied to growth processes [22, 23], the longest increasing subsequence of random permutations [24] and the theory of random partitions [25, 26].

There are several books and reviews on Random Matrix Theory that should be mentioned. The classic text is the book of Mehta [4] which emphasizes the orthogonal polynomial method. The supersymmetric method is discussed in detail in the book by Efetov [6]. In the book by Forester [27] the emphasis is on the relation with solvable models and mathematical physics. The historical developments of Random Matrix Theory are discussed in the comprehensive review by Guhr, Müller-Groeling, and Weidenmïler. [28]. A short historical overview of Random Matrix Theory can be found in the special edition

of Journal of Physics A which contains a broad selection of recent papers on Random Matrix Theory as well as several reviews [29]. Applications to mesoscopic physics are reviewed in [30], applications to QCD in [31] and applications to Quantum Gravity in [32]. Among the pedagogical reviews we mention a general review of Random Matrix Theory [33], an introduction to the supersymmetric method [34, 35] and applications of random matrix theory to quantum gravity [36].

2. RANDOM MATRIX THEORY

In this section we give an elementary introduction to Random Matrix Theory. Some of the material in the subsections below is based on the encyclopedia article [33].

2.1. The Wigner-Dyson ensembles

The Wigner-Dyson ensembles are ensembles of hermitian matrices with matrix elements distributed according to

$$P(H)DH = \mathcal{N} e^{-\frac{N\beta}{4}\text{Tr} H^{\dagger}H} DH.$$ (1)

Here, H is a Hermitian $N \times N$ matrix with real, complex, or quaternion real matrix elements, and β is the Dyson index of the ensemble. Its value is one for real matrix elements, two for complex matrix elements and four for quaternion real matrix elements. Real quaternions can be expressed in terms of the Pauli spin matrices as

$$q = a_0 + i a_k \sigma_k,$$ (2)

where the a_μ are real variables. The measure DH is the product over independent differentials. The normalization constant of the probability distribution is denoted by $\mathcal{N}$. The probability distribution is invariant under the transformations

$$H \to UHU^{-1},$$ (3)

where U is an orthogonal transformation for $\beta = 1$, a unitary transformation for $\beta = 2$, and a symplectic transformations for $\beta = 4$. This is the reason why these ensembles are known as the Gaussian Orthogonal Ensemble (GOE), the Gaussian Unitary Ensemble (GUE), and the Gaussian Symplectic Ensemble (GSE), respectively. Since both the eigenvalues of H and the Haar measure DH are invariant with respect to (3), the eigenvectors and the eigenvalues are independent with the distribution of the eigenvectors given by the invariant measure of the corresponding orthogonal, unitary, or symplectic group. Notice that eigenvalues of a quaternion real matrix are quaternion scalars and are doubly degenerate.

Exercise. Show that the symplectic ensemble is invariant under symplectic similarity transformations. A symplectic transformation is a unitary transformation that leaves the

matrix Z invariant where, in quaternion notation, Z is the direct product of $i\sigma_2$ and the identity.

Exercise. Show that the measure, dH, is invariant with respect to the similarity transformations (3). Prove this for each value of β.

One way of deriving the probability distribution (1) is to look for a distribution that minimizes the information entropy under the constraints that the average, variance and normalization of the probability distribution are fixed. The proof goes as follows [37]. The sum of the information entropy and the constraints is given by

$$S = \int DH[-P(H)\log P(H) + aP(H) + bP(H)\mathrm{Tr}H + cP(H)\mathrm{Tr}H^2]. \tag{4}$$

Minimizing this with respect to $P(H)$ we obtain

$$-1 - \log P(H) + a + b\mathrm{Tr}H + c\mathrm{Tr}H^2 = 0, \tag{5}$$

resulting in a Gaussian distribution for $P(H)$. The maximum entropy approach has also been successfully applied to the joint probability distribution of S-matrix elements [38] and the transport properties of quantum dots (see [39, 30]).

The joint probability distribution of the Wigner-Dyson ensembles is given by

$$P(\{\lambda_k\})\prod_k d\lambda_k = \mathscr{N}|\Delta(\{\lambda\})|^\beta \prod_k e^{-N\beta\lambda_k^2/4}d\lambda_k, \tag{6}$$

where the Vandermonde determinant is defined by

$$\Delta(\{\lambda\}) = \prod_{k>l}(\lambda_k - \lambda_l). \tag{7}$$

The Vandermonde determinant originates from the Jacobian of the transformation from the matrix elements as integration variables to the eigenvalues and eigenvectors as integration variables. The distribution of the eigenvectors completely factorizes from the probability distribution. As an example, let us calculate the Jacobian for $\beta = 2$. One way to do this is to consider the invariant distance

$$\mathrm{Tr}\, dH dH^\dagger. \tag{8}$$

By differentiating (3) we find

$$U^{-1}dHU = \delta U\Lambda - \Lambda\delta U + d\Lambda, \tag{9}$$

where $\delta U = U^{-1}dU$. For the invariant distance we then obtain

$$\begin{aligned}
\mathrm{Tr}\, dH dH^\dagger &= 2\mathrm{Tr}\delta U\Lambda\delta U\Lambda - 2\mathrm{Tr}\delta U\Lambda^2\delta U + \mathrm{Tr}d\Lambda^2 \\
&= 2\sum_{k<l}\delta U_{kl}\delta U_{kl}^*(\Lambda_k - \Lambda_l)^2 + \sum_k(d\Lambda_k)^2.
\end{aligned} \tag{10}$$

We can immediately read off the Jacobian from the Lamé coefficients and the result is given by the square of the Vandermonde determinant.

Exercise. Calculate the Jacobian for $\beta = 1$ and $\beta = 4$.

2.2. The chiral ensembles

A second class of ensembles are the chiral ensembles with the chiral symmetry of the QCD Dirac operator [17, 18]. They are defined as the ensembles of $N \times N$ Hermitian matrices with block structure

$$D = \begin{pmatrix} 0 & C \\ C^\dagger & 0 \end{pmatrix} \tag{11}$$

and probability distribution given by

$$P(C)DC = \mathcal{N} \det^{N_f} \begin{pmatrix} 0 & C \\ C^\dagger & 0 \end{pmatrix} e^{-\frac{N\beta}{4} \mathrm{Tr} C^\dagger C} DC. \tag{12}$$

Again, DC is the product of differentials of the independent parts of the matrix elements of C, and N_f is a real parameter (corresponding to the number of quark flavors in QCD). The matrix C is a rectangular $n \times (n+\nu)$ matrix. The nonzero eigenvalues of the matrix D occur in pairs $\pm\lambda_k$. This can be seen as follows. If

$$\begin{pmatrix} 0 & C \\ C^\dagger & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}, \tag{13}$$

then

$$\begin{pmatrix} 0 & C \\ C^\dagger & 0 \end{pmatrix} \begin{pmatrix} a \\ -b \end{pmatrix} = -\lambda \begin{pmatrix} a \\ -b \end{pmatrix}. \tag{14}$$

Generically, the matrix D in (11) has exactly $|\nu|$ zero eigenvalues. Also generically, the QCD Dirac operator corresponding to a field configuration with the topological charge ν has exactly $|\nu|$ zero eigenvalues, in accordance with the Atiyah-Singer index theorem. For this reason, ν is identified as the topological quantum number. The normalization constant of the probability distribution is denoted by $\mathcal{N}$. Also in this case one can distinguish ensembles with real, complex, or quaternion real matrix elements. They are denoted by $\beta = 1$, $\beta = 2$, and $\beta = 4$, respectively. The unitary invariance of the chiral ensembles is given by

$$C \rightarrow UCV^{-1}, \tag{15}$$

where U and V are orthogonal, unitary, or symplectic matrices, respectively. For this reason, the corresponding ensembles are known as the chiral Gaussian Orthogonal Ensemble (chGOE), the chiral Gaussian Unitary Ensemble (chGUE), and the chiral Gaussian Symplectic Ensemble (chGSE), in this order.

Exercise. Show that the eigenvalues of the matrix with structure (11) occur in pairs $\pm\lambda$ or are zero. If C is an $n \times (n+\nu)$ block matrix show that there are exactly ν zero eigenvalues.

It is always possible to decompose the matrix C in (12) as

$$C = U\Lambda V^{-1}, \tag{16}$$

where Λ is a diagonal matrix with $\Lambda_{kk} \geq 0$, and U and V are orthogonal matrices for $\beta = 1$, unitary matrices for $\beta = 2$ and symplectic matrices for $\beta = 4$. The joint probability distribution for the eigenvalues is obtained by transforming to Λ, U and V as new integration variables. The Jacobian can be obtained in the same way as in the previous section and is given by

$$J \sim \prod_k \lambda_k^{\beta\nu+\beta-1} \prod_{k<l} |(\lambda_k^2 - \lambda_l^2)|^\beta \tag{17}$$

resulting in the joint eigenvalue distribution

$$P(\{\lambda\})d\{\lambda\} = \mathcal{N}|\Delta(\{\lambda^2\})|^\beta \prod_k \lambda_k^\alpha e^{-N\beta\lambda_k^2/4} d\lambda_k, \tag{18}$$

where $\alpha = \beta - 1 + \beta\nu + 2N_f$. As in the case the Wigner-Dyson ensembles the distribution of the eigenvectors and the distribution of the eigenvalues factorize.

Exercise. Find the Jacobian (17) using only symmetry arguments and dimensional arguments.

2.3. The superconducting ensembles

A third class of random matrix ensembles occurs in the description of disordered superconductors [40, 41]. Such ensembles with the symmetries of the Bogoliubov-de Gennes Hamiltonian have the block structure

$$H = \begin{pmatrix} A & B \\ B^\dagger & -A^T \end{pmatrix}, \tag{19}$$

where A is Hermitian and, depending on the underlying symmetries, the matrix B is symmetric or anti-symmetric. The probability distribution is given by

$$P(H)DH = \mathcal{N}\exp\left(-\frac{N\beta}{4}\mathrm{Tr}H^\dagger H\right)DH, \tag{20}$$

where DH is the Haar measure and $\mathcal{N}$ is a normalization constant. For a symmetric matrix B the matrix elements of H can be either complex ($\beta = 2$) or real ($\beta = 1$). One can easily verify that in this case H satisfies the relation

$$H^T = \Gamma_A H \Gamma_A, \tag{21}$$

where

$$\Gamma_A = \begin{pmatrix} 0 & 1_N \\ -1_N & 0 \end{pmatrix}. \tag{22}$$

One can then show that if ϕ is an eigenvector of H with eigenvalue λ, then $\Gamma_A\phi$ is an eigenvector of H^T with eigenvalue $-\lambda$. Because H and H^T have the same eigenvalues, all eigenvalues occur in pairs $\pm\lambda$.

For an anti-symmetric matrix B the matrix elements of H can be either complex ($\beta = 2$) or quaternion real ($\beta = 4$). In that case H satisfies the relation

$$H^T = -\Gamma_S H \Gamma_S, \tag{23}$$

where

$$\Gamma_S = \begin{pmatrix} 0 & 1_N \\ 1_N & 0 \end{pmatrix}. \tag{24}$$

From this relation it follows that if ϕ is an eigenvector of H with eigenvalue λ, then $\Gamma_S \phi$ is an eigenvector of H^T with eigenvalue $-\lambda$. Also in this case, all eigenvalues occur in pairs $\pm\lambda$.

The joint eigenvalue probability distribution of these so called Oppermann-Altland-Zirnbauer ensembles has the same general form as the joint probability distribution chiral ensembles given in (18). The value of $\alpha = 1$ for $\beta = 1$ and $\beta = 4$ whereas for $\beta = 2$ the value of α is 0 or 2.

Exercise. Derive the joint eigenvalue distribution for the Oppermann-Altland-Zirnbauer ensembles.

2.4. Anti-Unitary symmetries and the Dyson index

The value of the Dyson index is determined by the anti-unitary symmetries of the system. If there are no anti-unitary symmetries the Hamiltonian is Hermitian and the value of $\beta = 2$.

An anti-unitary symmetry operator, which can always be written as $A = UK$ with U unitary and K the complex conjugation operator, commutes with the Hamiltonian of the system

$$[H, UK] = 0. \tag{25}$$

We can distinguish two possibilities

$$(UK)^2 = 1 \quad \text{or} \quad (UK)^2 = -1. \tag{26}$$

The argument goes as follows. The symmetry operator $A^2 = (UK)^2 = UU^*$ is unitary, and in an irreducible subspace of the unitary symmetries, it is necessarily a multiple of the identity, $UU^* = \lambda \mathbf{1}$. Because of this relation, U and U^* commute so that λ is real. By unitarity we have $|\lambda| = 1$ which yields $\lambda = \pm 1$.

The case $(UK)^2 = 1$ corresponds to $\beta = 1$ and the case $(UK)^2 = -1$ to $\beta = 4$. In the first case it is always possible to find a basis in which the Hamiltonian is real. Starting with basis vector ϕ_1 we construct $\psi_1 = \phi_1 + UK\phi_1$. Then choose ϕ_2 perpendicular to ψ_1 and define $\psi_2 = \phi_2 + UK\phi_2$. Then

$$(\phi_2 + UK\phi_2, \psi_1)$$

$$\begin{aligned}
&= (UK\phi_2, \psi_1) \\
&= ((UK)^2\phi_2, UK\psi_1)^* \\
&= (\phi_2, \psi_1)^* = 0.
\end{aligned} \tag{27}$$

The next basis vector is found by choosing ϕ_3 perpendicular to ψ_1 and ψ_2, etc.. In this basis the Hamiltonian is real

$$\begin{aligned}
H_{kl} &= (\psi_k, H\psi_l) \\
&= (UK\psi_k, UKH\psi_l)^* \\
&= (\psi_k, HUK\psi_l)^* \\
&= (\psi_k, H\psi_l)^* = H_{kl}^*.
\end{aligned} \tag{28}$$

The best known anti-unitary operator in this class is the time-reversal operator for which U is the identity matrix.

In the case $(UK)^2 = -1$ all eigenvalues of the Hamiltonian are doubly degenerate. This can be shown as follows. If ϕ_k is and eigenvector with eigenvalue λ_k, then it follows from the commutation relations that also $UK\phi_k$ is an eigenvector of the Hamiltonian with the same eigenvalue. The important thing is that this eigenvector is perpendicular to ϕ_k,

$$(\phi_k, UK\phi_k) = (UK\phi_k, (UK)^2\phi_k)^* = -(\phi_k, UK\phi_k). \tag{29}$$

One can prove that in this case it is possible to chose a basis for which the Hamiltonian matrix can be organized into real quaternions [42]. The eigenvalues of a Hermitian quaternion real matrix are quaternion scalars, and the eigenvalues of the original matrix are thus doubly degenerate in agreement with (29). The best known example in this class is the Kramers degeneracy for time reversal invariant systems of half-integral spin but no rotational invariance. Then the time reversal operator is given by $\sigma_2 K$ with $(\sigma_2 K)^2 = -1$.

Later in these lectures we will discuss other anti-unitary symmetries that enter in the Dirac operator for non-abelian Yang-Mills fields. They determine the Dyson index of the chiral ensembles.

2.5. Classification of the Random Matrix Ensembles

The Random Matrix Ensembles discussed above can be classified according to the Cartan classification of symmetric spaces [43, 44, 45]. A symmetric space is a manifold such that every point is a fixed point of an involutive isometry (i.e. $x_i \to -x_i$). The Riemann curvature tensor is covariantly constant in a symmetric space. A symmetric space is best characterized via the notion of symmetric pair. A symmetric pair (G, H) is defined as a pair of a connected Lie group G and a closed subgroup H such that an involutive analytic automorphism σ of G exists with $H \in H_\sigma$, where H_σ is the set of fixed points of σ. Then, with some additional conditions (see the book by Helgason [46] for more details) the coset G/H is a symmetric space.

As an example, consider the group $U(p+q)$ and define Γ_{pq} as the diagonal matrix with the first p diagonal matrix elements equal to 1 and the remaining q diagonal matrix

elements equal to -1. Consider the transformation

$$\sigma(g) = \Gamma_{pq} g \Gamma_{pq}, \tag{30}$$

which is an involution because obviously $\sigma(\sigma(g)) = g$. One can also easily show that it is an analytic automorphism. The set of fixed points of σ is the group $U(p) \times U(q)$. Therefore, $U(p+q)/U(p) \times U(q)$ is a Riemannian symmetric space.

A symmetric space can be of the compact, noncompact or the Euclidean type with positive, negative or zero curvature, respectively. Each of the random matrix ensembles discussed in this section is tangent to one of the large classes of symmetric spaces. The complete classification is given in table I [44] where the corresponding symmetric space of the compact type is also given.

TABLE 1. Random Matrix Ensemble, Corresponding symmetric space and the value of the Dyson index β.

RMT	symmetric space	group structure	β
GOE	AI	$U(N)/O(N)$	1
GUE	A	$U(N)$	2
GSE	AII	$U(2N)/Sp(N)$	4
chGOE	BDI	$SO(p+q)/SO(p) \times SO(q)$	1
chGUE	AIII	$U(p+q)/U(p) \times U(q)$	2
chGSE	CII	$Sp(p+q)/Sp(p) \times Sp(q)$	4
AZ-CI	CI	$Sp(N)/U(N)$	1
AZ-D	D	$SO(N)$	2
AZ-C	C	$Sp(N)$	2
AZ-DIII	DIII	$SO(2N)/U(N)$	4

For example, for special unitary matrices we can write $U = 1 + iH + \cdots$. Therefore, the Hermitian matrices are tangent to the space A (after dividing out a $U(1)$ factor). As another example, the generators $U(p+q)/U(p) \times U(q)$ (class AII) are given by matrices with the structure (11). More examples can be found in the recent review by Caselle and Magnea [47].

3. SPECTRA OF COMPLEX SYSTEMS

In this section we discuss quantum spectra and several frequently used statistics for the statistical analysis of eigenvalues.

3.1. General Remarks

Typically, when we talk about spectra we think about quantum mechanics. Among the best known examples are the spectrum of a harmonic oscillator with eigenvalues

$$E_n = (n + \frac{1}{2})\hbar\omega, \tag{31}$$

and the spectrum of an angular momentum operator with eigenvalues of J^2 given by

$$\hbar^2 j(j+1). \tag{32}$$

However, it is the exception that we are able to write down analytical formulas for the complete set of eigenvalues of an operator or even a single eigenvalue. One of the questions we wish to address is whether we are able to find any generic features in spectra even if we do not have any detailed analytical knowledge of the eigenvalues.

Let me first give examples of different systems for which spectra play an important role. One of the most important examples is the Schrödinger equation

$$(-\nabla^2 + V(x))\psi = \lambda\,\psi, \tag{33}$$

of course with boundary conditions on the wave functions. One particularly simple class of potentials are billiards where $V(x)$ is zero inside a connected domain and infinite outside this domain. In particular, billiards in the two-dimensional plane have been studied extensively in the literature. Although it is trivial to solve the Schrödinger equation, the complication arises because of the boundary conditions which can only be satisfied for a discrete set of eigenvalues.

A second example is the Helmholtz equation for the electric and magnetic field inside a resonance cavity

$$(\nabla^2 + k^2)\vec{E} = 0, \qquad (\nabla^2 + k^2)\vec{B} = 0. \tag{34}$$

In this case one can distinguish different types of boundary conditions, transverse electromagnetic, transverse magnetic and transverse electric. One case of special interest are transverse magnetic modes in a two-dimensional cavity. For E_z we find the wave equation

$$(\nabla^2 + k^2)E_z = 0, \tag{35}$$

with the boundary condition that E_z vanishes on the boundary. This is exactly the Schrödinger equation for a two-dimensional billiard. This equivalence has been exploited for the experimental analysis of quantum billiards [48].

A third example is that of sound waves in materials. In that case the wave equation is given by

$$(\nabla^2 + k^2)\vec{u} = 0, \tag{36}$$

where $\vec{u}$ can be decomposed in the longitudinal and the transverse displacement. This equation is more complicated than either the Maxwell or Schrödinger equation because the two types of modes are coupled through reflections at the boundary. Also this system has been a valuable tool for the experimental analysis of spectra of complex systems [49].

A fourth example is the Dirac operator in QCD. In this case the wave equation is given by

$$\gamma_\mu(\partial_\mu + iA_\mu)\psi = i\lambda\,\psi. \tag{37}$$

Here, the γ_μ are the Euclidean gamma matrices and the A_μ are $SU(N_c)$ valued gauge fields. We will discuss this example in much more detail later in these lectures.

As a fifth example I mention the zeros of the Riemann ζ function. This function is defined as

$$\zeta(s) = \sum_n \frac{1}{n^s}. \tag{38}$$

Empirically, it is know that all zeros are on the line $\mathrm{Re}\, s = \frac{1}{2}$ (also known as the Riemann conjecture). An intriguing question is what these zeros have in common with the eigenvalues of a complex dynamical system (for answers we refer to [21]).

All differential operators we have discussed up to now are Hermitian operators. They can be diagonalized by a unitary transformation and have real eigenvalues, i.e.

$$H = U\Lambda U^{-1}. \tag{39}$$

Later we will see applications of non-hermitian differential operators. They can be diagonalized by a similarity transformation with eigenvalues scattered in the complex plane.

In many cases spectra can be observed experimentally. The best-known examples are atomic and nuclear spectra. However, a great deal of spectroscopy has been performed on molecular spectra as well. As a more recent example I mention quantum dots in which electrons are enclosed inside a heterostructure and collide elastically against the boundary. Other examples are the spectra of small metallic particles, which are determined by the boundary conditions on the surface. In this case the specific heat can be related to the spectral properties of the particles. For references and an extensive discussion of these and other examples we refer the [28].

3.2. Statistical analysis of spectra

Although it is the exception that quantum spectra can be obtained analytically, it is often possible to obtain a large number of different eigenvalues either experimentally or by numerical calculations. This cries out for a reduction of information, and it is natural the perform a statistical analysis of the spectra. The average spectral density can be written as

$$\rho(\lambda) = \left\langle \sum_k \delta(\lambda - \lambda_k) \right\rangle \tag{40}$$

with normalization given by (for a finite Hilbert space)

$$\int d\lambda \rho(\lambda) = N. \tag{41}$$

The average can be either over an ensemble of different systems (ensemble average) or over different parts of a spectrum (spectral average). Of course it is also possible that the average is a mixture of both types of averages.

One frequently employs the integrated spectral density

$$N(E) = \int_{-\infty}^{E} \rho(\lambda)d\lambda. \tag{42}$$

Since $N(E)$ jumps by one at the position of each eigenvalue, it is also know as the staircase function.

Typically, the spectral density varies little over the scale of the average level spacing. Therefore the average spectral density can be obtained by locally averaging over an interval much larger than the average level spacing. Since $\langle \rho(\lambda) \rangle$ depends on the specific properties of the system one would like to eliminate this dependence. This is achieved by the so-called unfolding procedure. The unfolded spectrum is given by

$$\lambda_k^{\text{unf}} = \int_0^{\lambda_k} \langle \rho(\lambda) \rangle d\lambda. \tag{43}$$

One can easily verify that the average spacing of the unfolded sequence is equal to unity.

The statistical analysis of the unfolded eigenvalues can be performed by a variety of statistics. The simplest statistic is the so called nearest neighbor spacing distribution. It is denoted by $P(S)$ and is just a histogram of neighboring unfolded levels. A second class of statistics is obtained from counting the number of levels in q intervals along the spectrum that contains n levels on average. If the number of levels in consecutive intervals is given by n_k, we can define the moments

$$M_p(n) = \frac{1}{q} \sum_{k=1}^{q} n_k^p. \tag{44}$$

Of course, we have that $M_1(n) \to n$ for a large sample. The number variance is defined by

$$\Sigma^2(n) = M_2(n) - n^2. \tag{45}$$

Similarly, one can define higher order cumulants.

Another frequently used statistic is the Δ_3 statistic which is defined as

$$\Delta_3(L) = \frac{2}{L^4} \int_0^L (L^3 - 2L^2 r + r^3)\Sigma^2(r)dr. \tag{46}$$

It has the property that quadratic functions are projected to zero by the kernel of this integral operator. Generally, this statistic is much smoother than the number variance and for that reason it has been used widely in the literature.

Exercise. Calculate the number variance and the Δ_3 statistic for a picket fence spectrum, i.e. a spectrum of equally spaced eigenvalues.

3.3. Statistics for uncorrelated eigenvalues

Let us calculate $P(S)$ and $\Sigma_2(n)$ for independently distributed eigenvalues with average level spacing equal to unity. Then $P(S)dS$ is the probability that there are no

eigenvalues inside the interval $[0, S]$ and one eigenvalue in $[S, S + dS]$. Dividing the first interval in n equal pieces we find

$$P(S)dS = \left(1 - \frac{S}{n}\right)^n DS \to e^{-S}dS \quad \text{for} \quad n \to \infty. \tag{47}$$

The number variance can be expressed as

$$\begin{aligned}
\Sigma^2(n) &= \langle \int_0^n \rho(\lambda)d\lambda \int_0^n \rho(\lambda')d\lambda' \rangle - n^2, \\
&= \int_0^n \int_0^n d\lambda d\lambda' \langle \sum_{k,l} \delta(\lambda - \lambda_k)\delta(\lambda' - \lambda_l)\rangle - n^2,
\end{aligned} \tag{48}$$

where the λ_k are the unfolded eigenvalues. The average factorizes for different eigenvalues. For a finite number of levels we find

$$\Sigma^2(n) = \int_0^n \int_0^n d\lambda d\lambda' [\delta(\lambda - \lambda') + \frac{N(N-1)}{N^2} \sum_{k,l} \langle \delta(\lambda - \lambda_k)\rangle\langle \delta(\lambda' - \lambda_l)\rangle] - n^2. \tag{49}$$

The first term in the brackets is the diagonal contribution with $k = l$, and in the second term we have corrected the absence of the diagonal terms in the sum by a factor $N(N-1)/N^2$. After integration over λ and λ' we obtain

$$\Sigma_2(n) = n - \frac{n^2}{N}, \tag{50}$$

where N is the total number of eigenvalues.

The Δ_3-statistic is obtained by a simple integration resulting in

$$\Delta_3(L) = \frac{L}{15} + O(1/N). \tag{51}$$

However, if one calculates $P(S)$ and $\Sigma_2(n)$ for spectra of interacting systems one finds different results. In particular, one notices a suppression of small spacings and a strongly reduced number variance for large n. These properties are characteristic for spectra of many complex systems and are known as level repulsion and spectral rigidity, respectively.

3.4. Correlation functions

The number variance can be expressed in terms of the two-point correlation function of the unfolded eigenvalues. The two-point correlation function of the not necessarily unfolded eigenvalues is defined by

$$\rho_2(\lambda, \lambda') = \langle \rho(\lambda)\rho(\lambda')\rangle - \langle \rho(\lambda)\rangle\langle \rho(\lambda')\rangle. \tag{52}$$

The nearest neighbor spacing distribution cannot be expressed in terms a two-point correlator only. The reason is that it measures the probability that none of the other eigenvalues are inside the interval $[0, S]$.

The correlation function $\rho_2(\lambda, \lambda')$ includes a term in which the eigenvalues are equal, and can thus be decomposed as

$$\rho_2(\lambda, \lambda') = \delta(\lambda - \lambda')\langle\rho(\lambda)\rangle + R_2(\lambda, \lambda'). \tag{53}$$

The two-point correlation function satisfies the sum rule

$$\int d\lambda \rho_2(\lambda, \lambda') = 0, \tag{54}$$

where the integral is over the complete spectrum.

In the literature one also frequently uses the quantity $Y_2(\lambda, \lambda')$ for the two-point correlation function of the unfolded eigenvalues [4]

$$Y_2(\lambda, \lambda') = -R_2(\lambda, \lambda'), \tag{55}$$

where the minus sign is conventional. If a smoothened average spectral density exists, it is natural to expect that Y_2 is translational invariant, i.e.,

$$Y_2(\lambda, \lambda') = Y_2(\lambda - \lambda'). \tag{56}$$

In that case, the number variance can be expressed as

$$\Sigma^2(L) = L - \int_0^L (L - r)Y_2(r)dr. \tag{57}$$

4. THE SUPERSYMMETRIC METHOD IN RANDOM MATRIX THEORY

The supersymmetric has been introduced in Random Matrix Theory and the theory of disordered systems to calculate quenched averages according to [7, 6, 50]

$$\left\langle \text{Tr}\frac{1}{z+D} \right\rangle = \partial_z \left\langle \frac{\det(D+z)}{\det(D+z')} \right\rangle \Bigg|_{z'=z}. \tag{58}$$

The determinant can be expressed as a fermionic integral and the inverse determinant as a bosonic integral. For $z' = z$ the generating function has an exact supersymmetry. In this lecture we will discuss in detail the supersymmetric method. We start with the introduction of Grassmann variables and Grassmann integration. Our main objects are graded vectors and graded matrices that consist of both bosonic and fermionic variables. As application we will discuss the one-point function and the two-point function of the Gaussian Unitary Ensemble. For more details we refer to the book by Efetov [6] and [9] where the class $\beta = 1$ is worked out in detail.

4.1. Definitions

Grassmann variables are anti-commuting variables

$$\{\chi_k, \chi_l\} = 0, \tag{59}$$

which are necessarily nilpotent, i.e. $\chi_k^2 = 0$. One can introduce two types of conjugation, namely, conjugation of the first kind with the property that

$$\chi^{**} = \chi, \tag{60}$$

and conjugation of the second kind with the property

$$\chi^{**} = -\chi. \tag{61}$$

Below it will become clear that it is often advantageous to use conjugation of the second kind.

The integration over Grassmann variables is defined by

$$\int d\chi = 0, \qquad \int \chi d\chi = 1, \tag{62}$$

and thus more resembles a differentiation than an integration. As an example, consider the integral

$$\int d\chi d\chi^* e^{-\chi a \chi^*} = \int d\chi d\chi^* (1 - \chi a \chi^*) = a. \tag{63}$$

Notice that also the differentials $d\chi$ are anti-commuting elements.

Next let us make a change of integration variables, $\xi = a\chi$. Then

$$1 = \int \xi d\xi = \int a\chi d(a\chi), \tag{64}$$

so that we have the relation

$$d(a\chi) = \frac{1}{a} d\chi. \tag{65}$$

For a general transformation given by $\xi_k = A_{kl}\chi_l$ we have that

$$d\xi_1 \cdots d\xi_n = \frac{1}{\det A} d\chi_1 \cdots \chi_n. \tag{66}$$

Exercise. Prove this relation.

An important class of integrals are Gaussian Grassmannian integrals. Let us consider

$$I \equiv \int d\chi_1 \cdots d\chi_n d\chi_1^* \cdots d\chi_n^* e^{-\chi_k A_{kl} \chi_l^*}. \tag{67}$$

Because of the nil-potency and the definition of the Grassmann integration we obtain

$$I \equiv \int d\chi_1 \cdots d\chi_n d\chi_1^* \cdots d\chi_n^* \frac{1}{n!} (-\chi_k A_{kl} \chi_l^*)^n.$$ (68)

By renaming the integration variables one easily finds that $n!$ of the terms contributing to the integral are the same. We thus find

$$I = \int d\chi_1 \cdots d\chi_n d\chi_1^* \cdots d\chi_n^* (-1)^n \sum_\pi \chi_1 A_{1\pi(1)} \chi_{\pi(1)}^* \cdots \chi_n A_{n\pi(n)} \chi_{\pi(n)}^*.$$ (69)

where the sum is over all permutations of $1, \cdots, n$. By rearranging the differential it easily follows that the result of Grassmann integrations is exactly the sign of the permutation, and

$$I = \det A.$$ (70)

This should be compared to a bosonic integral

$$\frac{1}{(-2\pi i)^n} \int d\phi_1 \cdots d\phi_n d\phi_1^* \cdots d\phi_n^* e^{-\phi_k A_{kl} \phi_l^*} = \frac{1}{\det A}.$$ (71)

Exercise. Prove this result.

An essential difference between fermionic and bosonic integrals is that fermionic integrals are always convergent, whereas bosonic integrals only exist for specific integration contours.

4.2. Graded vectors and Graded Matrices

An $(m|n)$ graded vector is defined by

$$\phi_k \equiv \begin{pmatrix} S_1 \\ \vdots \\ S_m \\ \chi_1 \\ \vdots \\ \chi_n \end{pmatrix}.$$ (72)

The matrices acting on these vectors have the structure

$$M = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix},$$ (73)

where a and b are matrices with commuting entries and ρ and σ are matrices with anti-commuting entries. The block structure is such that a is an $m \times m$ matrix and b is an $n \times n$ matrix.

The trace of a graded matrix is defined by

$$\text{Trg}M = \text{Tr}a - \text{Tr}b. \tag{74}$$

A graded trace of a product of graded matrices is invariant under cyclic permutations.

The graded determinant is defined by the relation

$$\text{detg}M = \exp\text{Trg}\log M = \exp\text{Trg}\log\left[\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}[1 + \begin{pmatrix} 0 & a^{-1}\sigma \\ b^{-1}\rho & 0 \end{pmatrix}]\right]. \tag{75}$$

The logarithm can be expanded in a power series and only the even powers contribute to the graded trace. Resumming the power series again into a logarithm, one finds

$$\text{detg}M = \frac{\det(a - \sigma b^{-1}\rho)}{\det b}. \tag{76}$$

Exercise. Show that $\text{detg}(AB) = \text{detg}A\text{detg}B$.

The transpose of the matrix M is defined by

$$M^T = \begin{pmatrix} a^T & \rho^T \\ -\sigma^T & B^T \end{pmatrix}. \tag{77}$$

Because of the additional minus sign we have the relation

$$(M_1 M_2)^T = M_2^T M_1^T. \tag{78}$$

The Hermitian conjugate is defined in the usual way

$$M^\dagger = M^{*T}. \tag{79}$$

A super-matrix is Hermitian if $M^\dagger = M$. For a scalar product the Hermitian conjugate satisfies the usual relation

$$(\phi_1, M\phi_2) = (M^\dagger \phi_1, \phi_2), \tag{80}$$

where we have used the conjugation of the second kind.

The eigenvalues of a supermatrix are defined by

$$M\phi_k = \lambda_k \phi_k. \tag{81}$$

For two different eigenvectors of a super-Hermitian matrix M we have

$$(\phi^k, M\phi^l) = (M\phi^k, \phi^l). \tag{82}$$

This can be rewritten as

$$(\lambda_k - \lambda_l)(\phi_k, \phi_l) = 0, \tag{83}$$

so that the eigenvectors corresponding to different eigenvalues are orthogonal. As a consequence, a super-Hermitian matrix can be diagonalized by a super-unitary matrix, where a super-unitary matrix is defined in the usual way

$$U^\dagger U = 1. \tag{84}$$

The eigenvalues of a graded matrix with block structure (104) are given by the zeros and the poles of the secular equation

$$\det g(Q - \lambda) = \frac{\det(a - \lambda - \sigma(b - \lambda)^{-1}\rho)}{\det(b - \lambda)} = 0. \tag{85}$$

The poles of the secular equation can be obtained from the equation

$$\det g\frac{1}{(Q - \lambda)} = \frac{\det(b - \lambda - \rho(a - \lambda)^{-1}\sigma)}{\det(a - \lambda)} = 0. \tag{86}$$

As an illustration, it is suggested to the reader to calculate the eigenvalues of a diagonal matrix.

The ordinary parts of the eigenvalues are obtained by putting σ and ρ equal to zero and are thus given by the eigenvalues of a and b. The eigenvalues with ordinary part equal to one of the eigenvalues of a are given by the solutions of eq. (85), and the eigenvalues with ordinary part equal to one of the eigenvalues of b are given by the solutions of eq. (86). Notice that eq. (85) is singular for λ equal to one of the eigenvalues of b.

Exercise. Show that the eigenvalues of a $(1|1)$ super-matrix

$$\begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix} \tag{87}$$

are given by

$$\begin{aligned} \lambda_1 &= a + \frac{\sigma\rho}{a - b}, \\ \lambda_2 &= b + \frac{\sigma\rho}{a - b}. \end{aligned} \tag{88}$$

Next, let us calculate the following Gaussian graded integral

$$I \equiv \frac{1}{(-2\pi i)^m} \int d\phi d\phi^* e^{-\phi_k^* M_{kl}\phi_l}, \tag{89}$$

where, for reasons of convergence, the matrix M is a super-Hermitian matrix of the form (73). In the block notation introduced earlier in this chapter, the exponent can be written as

$$\begin{aligned} \phi_k^* M_{kl}\phi_l &= S^\dagger aS + S^\dagger \sigma\chi + \chi^\dagger \rho S + \chi^\dagger b\chi \\ &= (S^\dagger + \chi^\dagger \rho a^{-1})a(S + a^{-1}\sigma\chi) + \chi^\dagger(b - \rho a^{-1}\sigma)\chi. \end{aligned} \tag{90}$$

After shifting the bosonic integration variables, the integral factorizes into a bosonic and fermionic piece which we already know how to do. The result is

$$I = \frac{\det(b - \rho a^{-1}\sigma)}{\det a} = \frac{\det b}{\det(a - \sigma b^{-1}\rho)} = \frac{1}{\det g M}. \tag{91}$$

The second identity follows by writing the determinant as an exponent of the trace of a logarithm, and perform a cyclic permutation of the graded trace. The minus sign resulting in the inverse determinant comes from the anti-commutation of the Grassmann variables.

It is now clear why the graded trace was introduced as in (74): A graded Gaussian integral is given by the inverse graded determinant.

Exercise. Calculate the superintegrals for the case that the Grassmannian blocks of M are equal to zero.

Finally, a work of caution about notation. In the physics literature "graded" and "super" are used as synonyms. For example, graded vectors and supervectors, graded matrices and supermatrices, etc. are synonyms. The same is true for abbreviations. For example, $\mathrm{Str} \equiv \mathrm{Trg}$, $\mathrm{Sdet} \equiv \mathrm{detg}$. We will use both notations interchangeably. In quantum field theory, supersymmetry is used for a Grassmannian extension of the Poincaré group, and is different from the use of supersymmetry in these lectures.

5. INTEGRATION THEOREMS

If an integrand is invariant under super-unitary transformations, there exist a number of powerful integration theorems. In character they are comparable to a complex contour integration.

5.1. The Parisi-Sourlas reduction

Let us first consider the simplest case involving the graded vectors

$$p = \begin{pmatrix} a \\ \theta \end{pmatrix}, \qquad p^\dagger = (a^*, \theta^*). \tag{92}$$

Let $F(p, p^\dagger)$ be an invariant function, i.e.,

$$F(Up, U^\dagger p^\dagger) = F(p, p^\dagger), \tag{93}$$

for an arbitrary super-unitary transformation U. In this case the integration theorem states that

$$\frac{1}{2\pi} \int da\, da^*\, d\theta\, d\theta^* F(p, p^\dagger) = 2i F(0, 0), \tag{94}$$

which is also known as Parisi-Sourlas dimensional reduction [51]. To prove this theorem, we expand F in powers of the Grassmann variables (see also Appendix L of [9])

$$F(p, p^\dagger) = F_{00} + F_{01}\theta^* + F_{10}\theta + F_{11}\theta^*\theta. \tag{95}$$

If we choose

$$u = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}, \tag{96}$$

then the r.h.s. is only invariant if $F_{01} = F_{10} = 0$. By performing an infinitesimal transformation

$$u = \begin{pmatrix} 1 & \alpha \\ \alpha^* & 1 \end{pmatrix}, \tag{97}$$

we find that

$$F_{11} = \frac{1}{a}\partial_{a^*}F_{00}. \tag{98}$$

Exercise. Show that F_{00} is a function of aa^*.

After performing the Grassmann integration, the integral (94) is given by

$$\frac{1}{2\pi}\int dada^*\frac{1}{a}\partial_{a^*}F_{00}. \tag{99}$$

In polar coordinates the integral can be rewritten as

$$\frac{-2i}{2\pi}\int \rho d\rho d\phi\frac{1}{\rho}\partial_\rho F_{00}. \tag{100}$$

For a function that is well-behaved at infinity we then find that the integral is given by $2iF(0,0)$.

This theorem can be extended to supervectors with an arbitrary, but equal number of commuting and anticommuting components. We leave it up to the reader to formulate this theorem.

5.2. The Wegner Theorem

Next we consider an integration theorem for integrals over invariant functions of a supermatrix. This theorem was formulated and proved in lectures by Wegner [52] but appeared in published form six years later by different authors [53]. In this case the invariant function satisfies the relation

$$F(Q) = F(UQU^{-1}), \tag{101}$$

where U is a super-unitary matrix. The general form of an invariant function is thus given by

$$F(Q) = \sum_k a_k \text{Trg} Q^k. \tag{102}$$

Therefore, automatically, F is also invariant under general super-linear transformations. The theorem states that

$$\int DQ F(Q) = cF(\lambda \mathbf{1}), \tag{103}$$

where λ is an arbitrary constant Notice that this constant is not determined for an invariant function. The constant c depends on the size of the matrix Q and the choice of the integration contour. In the example to be discussed below its value is equal to i.

We first prove this theorem for the simplest case of a $(1|1)$ matrix. In this case, any function of the graded matrix

$$Q = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix} \tag{104}$$

can be expanded as

$$F(Q) = F_{00} + \sigma F_{01} + \rho F_{10} + \sigma \rho F_{11}. \tag{105}$$

If we choose

$$U = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{i\theta} \end{pmatrix}, \tag{106}$$

Then,

$$UQU^{-1} = \begin{pmatrix} a & \sigma e^{i\phi - i\theta} \\ \rho e^{i\theta - i\phi} & b \end{pmatrix}. \tag{107}$$

Invariance of $F(Q)$ then requires that $F_{10} = F_{01} = 0$ in the expansion (105). Next consider the invariance under

$$U = \begin{pmatrix} 1 & \omega \\ \zeta & 1 \end{pmatrix}. \tag{108}$$

Then to first order in ω and ζ

$$UQU^{-1} = \begin{pmatrix} a + \omega\rho + \zeta\sigma & \omega(b-a) + \sigma \\ \zeta(a-b) + \rho & b + \omega\rho + \zeta\sigma \end{pmatrix}. \tag{109}$$

From the invariance of F we then find

$$(\omega\rho + \zeta\sigma)(\partial_a F_{00} + \partial_b F_{00}) + (b-a)(\omega\rho + \zeta\sigma)F_{11} = 0. \tag{110}$$

This results in

$$F_{11} = \frac{\partial_a F_{00} + \partial_b F_{00}}{a - b}. \tag{111}$$

Example. Consider the invariant function $F(Q) = \text{Trg} Q^2$. In components we obtain $F(Q) = a^2 - b^2 + 2\sigma\rho$, which is in agreement with the above general result.

After performing the Grassmann integrals we find

$$\int DQ F(Q) = \frac{1}{2\pi} \int dadb \frac{\partial_a F_{00} + \partial_b F_{00}}{a - b}. \tag{112}$$

This integral is not well defined if a and b are along the same path in the complex plane. For definiteness let us take a along the real axis and b along the imaginary axis. In polar coordinates

$$a = \rho \cos\phi,$$
$$b = i\rho \sin\phi. \tag{113}$$

Then

$$\partial_a F_{00} + \partial_b F_{00} = (\frac{a-b}{\rho} \partial_\rho + \frac{a-b}{i\rho^2} \partial_\phi) F_{00}. \tag{114}$$

The integral over ϕ of the second term gives zero provided that the point $\rho = 0$ is suitably regularized. The integral over ρ of the first term is the integral of a total derivative. For well-behaved functions F we thus find

$$\int DQ F(Q) = iF(0) = iF(\lambda \mathbf{1}), \tag{115}$$

where the second equality follows from the expansion (102).

An inductive proof for the general case of this theorem was also given by Wegner [52]. He considered the case of an arbitrary $(m|n)$ matrix Q. In that case an integral over an $(m-n) \times (m-n)$ ordinary matrix remains in the final result (for $m > n$). However, for $m = n$, the theorem is exactly as in (115) but with Q equal to an $(m|n)$ super-matrix.

The proof of the theorem starts from the observation that an arbitrary $(m|n)$ matrix can be decomposed into the $(1|1)$ super-matrix

$$\bar{Q} = \begin{pmatrix} Q_{11} & Q_{1,n+m} \\ Q_{n+m,1} & Q_{n+m,n+m} \end{pmatrix}, \tag{116}$$

the $(1|1)$ super-vectors

$$P_\alpha = \begin{pmatrix} Q_{1\alpha} \\ Q_{n+m\alpha} \end{pmatrix}, \qquad \alpha = 2, \cdots n+m-1, \tag{117}$$

and

$$\bar{P}_\alpha = (Q_{\alpha 1}, Q_{\alpha n+m}), \qquad \alpha = 2, \cdots n+m-1. \tag{118}$$

The remaining matrix elements will be denoted by Q_R. We can thus write

$$F(Q) = \bar{F}(\bar{Q}, P, \bar{P}, Q_R). \tag{119}$$

We now consider the following invariance

$$\bar{F}(U\bar{Q}U^{-1}, UP, \bar{P}U^{-1}, Q_R) = \bar{F}(\bar{Q}, P, \bar{P}, Q_R). \tag{120}$$

and define

$$I(P, \bar{P}, Q_R) = \int D\bar{Q}\bar{F}(\bar{Q}, P, \bar{P}, Q_R). \tag{121}$$

By using the invariance of the measure one easily obtains that

$$I(UP, \bar{P}U^{-1}, Q_R) = I(P, \bar{P}, Q_R). \tag{122}$$

We thus can apply the Parisi-Sourlas theorem to the P and $\bar{P}$ integrations. Next we can apply Wegner's theorem for $(1|1)$ matrices to the $\bar{Q}$ integrations. What remains is an integral over the smaller super-matrix Q_R, thereby proving the general form of the theorem.

5.3. The Efetov-Zirnbauer theorem

In general, one is interested in super-integrals of a non-supersymmetric integrand. As an example we work out the $(1|1)$ case given by

$$\int DQ(a_{00}(Q) + a_{10}(Q)\sigma + a_{01}(Q)\rho + a_{11}(Q)\sigma\rho), \tag{123}$$

where Q is parametrized as in (104) and $a_{ij}(Q)$ are invariant functions. Because of the general decomposition of invariant functions, we immediately find that the integrals of the second and third terms are zero. However, as was observed by Zirnbauer, there is another way of deriving this result.

The proof by Zirnbauer [54] is based on the invariance of the measure. If we apply the transformation (109) to the Q variables in the integral we obtain

$$\int DQa_{10}(Q)\sigma\rho = \int DQa_{10}(Q)(\sigma + \omega(b-a))(\rho + \zeta(a-b)). \tag{124}$$

Since this equation is valid for arbitrary ω, we find that the second term in (123) integrates to zero. In the same way we find that the third term in (104) integrates to zero.

To better appreciate this theorem let us consider a more complicated integral involving an invariant function of $Q^\dagger Q$ with Q the $(1|1)$ supermatrix given in eq. (104). The matrix $Q^\dagger Q$ is then given by

$$Q^\dagger Q = \begin{pmatrix} a^*a + \rho^*\rho & a^*\sigma + b\rho^* \\ -a\sigma^* + b^*\rho & b^*b - \sigma^*\sigma \end{pmatrix}. \tag{125}$$

The eigenvalues of this matrix are given by

$$
\begin{aligned}
\lambda_1 &= a^*a + \rho^*\rho - \frac{a^*a\sigma^*\sigma + b^*b\rho^*\rho + a^*b^*\sigma\rho + ab\sigma^*\rho^*}{a^*a - b^*b} + \frac{(a^*a + b^*b)\sigma^*\sigma + \rho^*\rho}{(a^*a - b^*b)^2}, \\
\lambda_2 &= \lambda_1 - a^*a + b^*b - \sigma^*\sigma - \rho^*\rho.
\end{aligned}
\tag{126}
$$

A general invariant function of $Q^\dagger Q$ can therefore be decomposed as

$$
F(Q^\dagger Q) = F_{00} + F_{11}\sigma\rho + F_{11}^*\sigma^*\rho^* + F_{20}\sigma^*\sigma + F_{02}\rho^*\rho + F_{22}\sigma^*\sigma\rho^*\rho,
\tag{127}
$$

where the F_{ij} are functions of the commuting variables only. The invariance is given by

$$
Q \to UQV,
\tag{128}
$$

where U and V are super-unitary matrices.

Because of this decomposition the super-invariant integral over the product of an invariant function and an odd number of Grassmann variables is trivially zero. According to a theorem due to Zirnbauer [54] any super-invariant integral over the product of an invariant function and two Grassmann variables also vanishes. The proof proceeds as in our previous example and in this case we consider the infinitesimal transformation

$$
\begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix} \to \begin{pmatrix} 1 & \alpha \\ \beta & 1 \end{pmatrix} \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix} \begin{pmatrix} 1 & \zeta \\ \omega & 1 \end{pmatrix}.
\tag{129}
$$

and

$$
\begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix}^\dagger \to \begin{pmatrix} 1 & -\zeta \\ -\omega & 1 \end{pmatrix} \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix}^\dagger \begin{pmatrix} 1 & -\alpha \\ -\beta & 1 \end{pmatrix}.
\tag{130}
$$

If we apply this transformation to the integrand

$$
\int dQ\, F(Q^\dagger Q)\sigma\sigma^*\rho\rho^*,
\tag{131}
$$

and use the invariance of $F(Q^\dagger Q)$ and the invariance of the measure we find that the integral over the product of an invariant function and two Grassmann variables vanishes. Since this result was also used in Efetov's calculation [7], this theorem is known as the Efetov-Zirnbauer theorem.

One might think that the same argument can be applied to the integral of an invariant function. Obviously, because of Wegner's theorem this is not true. The reason is that the integral (131) diverges in this case. Typically, the invariant measure diverges if two eigenvalues coincide. In the next subsection we will analyze this singularity in detail.

5.4. Efetov-Wegner terms

Let us study the following $(1|1)$ super-invariant integral

$$
I = \frac{1}{\pi} \int da\, db\, d\sigma\, d\rho\, e^{-\mathrm{Tr}Q^\dagger Q},
\tag{132}
$$

where

$$Q = \begin{pmatrix} a & \sigma \\ \rho & ib \end{pmatrix}.$$ (133)

The additional factor i has been inserted so that the integral converges for the b-integration along the real axis. By expanding the Grassmann variables and performing the Gaussian integral one immediately finds that $I = 1$. Next we choose the eigenvalues of Q and σ and ρ as new integration variables. The Jacobian of the transformation from (λ_1, λ_2) to (a, b) is given by

$$J = \det \begin{pmatrix} 1 + \frac{\sigma\rho}{(a-ib)^2} & \frac{-i\sigma\rho}{(a-ib)^2} \\ \frac{\sigma\rho}{(a-ib)^2} & i - \frac{i\sigma\rho}{(a-ib)^2} \end{pmatrix} = i.$$ (134)

We thus find that our integral is given by

$$I = \int \frac{d\lambda_1 d\lambda_2 d\sigma d\rho}{\pi i} e^{-\lambda_1^2 + \lambda_2^2} = 0,$$ (135)

where the integral over λ_2 is along the imaginary axis. However, because of the Grassmann integration the result of the integral is zero. Obviously, something went wrong! The only place where we could have made an error is at the point $a = ib$, where the variable transformation becomes singular.

Let us do the integral by excluding an infinitesimal sphere around this point from the integration domain. This approach was followed by Haldane and Zirnbauer [55] and by Gossiaux, Pluhar and Weidenmüller [56]. In terms of the original variables this changes the integral by only an infinitesimal amount. Our integral is thus given by

$$I = \lim_{\varepsilon \to 0} I_\varepsilon$$ (136)

with

$$I_\varepsilon = \int dQ e^{-\text{Trg}Q^2} \theta(\sqrt{a^2 + b^2} - \varepsilon).$$ (137)

In terms of our new integration variables we have that

$$\sqrt{a^2 + b^2} = \sqrt{\lambda_1^2 - \lambda_2^2} - \frac{\sigma\rho}{\sqrt{\lambda_1^2 - \lambda_2^2}},$$ (138)

and our θ function is given by

$$\theta(\sqrt{a^2 + b^2} - \varepsilon) = \theta(\sqrt{\lambda_1^2 - \lambda_2^2} - \varepsilon) - \sigma\rho \frac{1}{\sqrt{\lambda_1^2 - \lambda_2^2}} \delta(\sqrt{\lambda_1^2 - \lambda_2^2} - \varepsilon).$$ (139)

Additional terms due to the δ-function are known as Efetov-Wegner terms. If we use polar coordinates for λ_1 and λ_2/i, the anomalous contribution to our integral can be

written as

$$\frac{1}{\pi} \int r dr d\phi d\sigma d\rho \frac{1}{r} \delta(r-\varepsilon) e^{-r^2} \rho\sigma = 1, \tag{140}$$

which is the correct answer for the Gaussian integral (132. In this case the non-anomalous piece is zero.

In general one is interested in non-superinvariant integrals of the following type

$$I = \int dQ(f_0(Q) + \rho\sigma f_1(Q)). \tag{141}$$

with f_0 and f_1 depending only of the eigenvalues of Q. In terms the new integration variables the integral is given by

$$I = f_0(0,0) + \frac{1}{2\pi i} \int d\lambda_1 d\lambda_2 f_1(\lambda_1, \lambda_2), \tag{142}$$

This result can also be interpreted in terms of the integration measure which is now given by

$$\frac{1}{2\pi i} d\lambda_1 d\lambda_2 d\sigma d\rho + \frac{1}{2\pi i} \frac{\delta(\sqrt{\lambda_1^2 - \lambda_2^2})}{\sqrt{\lambda_1^2 - \lambda_2^2}} d\lambda_1 d\lambda_2. \tag{143}$$

6. THE SUPERSYMMETRIC METHOD OF RANDOM MATRIX THEORY: THE ONE-POINT FUNCTION

In this chapter we calculate the one-point function of the GUE by means of the supersymmetric method. It is evaluated starting from the RMT formulation of the problem.

6.1. The One-Point Function

In this subsection we calculate the average resolvent for the Gaussian Unitary Ensemble by means of the supersymmetric method. This section closely follows the corresponding section in the article [33]. The starting point is

$$\langle G(z) \rangle = \frac{1}{N} \left\langle \text{Tr} \frac{1}{z-H} \right\rangle = \frac{1}{N} \frac{\partial}{\partial J} \bigg|_{J=0} Z(J), \tag{144}$$

where the generating function is defined by

$$Z(J) = \int DHP(H) \frac{\det(z-H+J)}{\det(z-H)}, \tag{145}$$

and the integral is over the probability distribution of one of the random matrix ensembles defined in the first chapter. The determinants can be expressed in terms of Gaussian integrals,

$$\frac{\det(z-H+J)}{\det(z-H)} = \int d\psi \exp\left(-\sum_{kl}[i\phi_k^*(z-H)_{kl}\phi_l + i\chi_k^*(z+J-H)_{kl}\chi_l]\right),$$

(146)

where the measure is defined by

$$d\psi = \prod_{j=1}^{N} \frac{d\phi_j d\phi_j^* d\chi_j d\chi_j^*}{-2\pi i}.$$

(147)

For convergence the imaginary part of z has to be positive.

Exercise. Show that that $Z(0) = 1$.

For the GUE (with Dyson index $\beta = 2$) the probability distribution is given by

$$P(H)DH = \mathcal{N} e^{-\frac{N}{2}\text{Tr}H^\dagger H} DH.$$

(148)

The Gaussian integrals over H can be performed trivially, resulting in the generating function

$$Z(J) = \int d\psi \exp\left[-\frac{1}{2N}\text{Trg}\left(\begin{array}{cc} \sum_j \phi_j^*\phi_j & \sum_j \chi_j^*\phi_j \\ \sum_j \chi_j\phi_j^* & \sum_j \chi_j^*\chi_j \end{array}\right)^2\right.$$
$$\left.-i\sum_j(\phi_j^* z\phi_j + \chi_j^*(z+J)\chi_j)\right],$$

(149)

where the sums over j run from 1 to N. The quartic terms in ϕ and χ can be expressed as Gaussian integrals by means of a Hubbard-Stratonovitch transformation. This results in

$$Z(J) = \int d\psi d\sigma \exp\left[-\frac{N}{2}\text{Trg}\,\sigma^2 - i\sum_j \left(\begin{array}{c}\phi_j^* \\ \chi_j^*\end{array}\right)^T (\sigma+\zeta)\left(\begin{array}{c}\phi_j \\ \chi_j\end{array}\right)\right],$$

(150)

where

$$\sigma = \left(\begin{array}{cc} \sigma_{BB} & \sigma_{BF} \\ \sigma_{FB} & i\sigma_{FF} \end{array}\right)$$

(151)

and

$$\zeta = \left(\begin{array}{cc} z & 0 \\ 0 & z+J \end{array}\right).$$

(152)

The variables σ_{BB} and σ_{FF} are commuting (bosonic) variables that range over the real axis. Both σ_{BF} and σ_{FB} are Grassmann (fermionic) variables. One easily verifies that

by completing the square and carrying out the σ-integrations one recovers the original quartic interaction.

The integrals over the ϕ and the χ variables are now Gaussian and can be performed trivially. This results in the σ-model

$$Z(J) = \int d\sigma \exp\left[-\frac{N}{2}\mathrm{Trg}\sigma^2 - N\mathrm{Trg}\log(\sigma+\zeta)\right]. \tag{153}$$

For $J = 0$ we deal with a super-invariant integral, and, according to Wegner's theorem, we have $Z(0) = 1$. By shifting the integration variables according to $\sigma \to \sigma - \zeta$ and carrying out the differentiation with respect to J one easily finds that

$$\langle G(z)\rangle = \langle z - i\sigma_{FF}\rangle. \tag{154}$$

In the large N limit, the expectation value of σ_{FF} follows from a saddle-point analysis. The saddle point equation for σ_{FF} is given by

$$\sigma_{FF} + iz = 1/\sigma_{FF} \tag{155}$$

resulting in the resolvent

$$\langle G(z)\rangle = \frac{z}{2} - \frac{i}{2}\sqrt{4-z^2}. \tag{156}$$

Using the relation between the resolvent and the spectral density we find that the average spectral density is a semi-circle.

6.2. Exact calculation of the One-Point Function

In fact, the integrals in (153) can be evaluated exactly [57] instead of performing a saddle point approximation. If we parameterize σ as

$$\sigma = \begin{pmatrix} a & \alpha \\ \beta & ib \end{pmatrix}, \tag{157}$$

the partition function can be rewritten as

$$Z(J) = \frac{1}{2\pi} \int dadbd\alpha d\beta \frac{(ib+J+z-\frac{\alpha\beta}{a+z})^N}{(a+z)^N} e^{-\frac{N}{2}(a^2+b^2+2\alpha\beta)}. \tag{158}$$

By shifting the integration variables and expanding the Grassmann variables, the integral can be rewritten as

$$Z(J) = \frac{1}{2\pi} \int dadbd\alpha d\beta \frac{(ib)^N}{a^N}(1-\frac{N\alpha\beta}{aib})(1-N\alpha\beta)e^{-\frac{N}{2}((a-z)^2+(b+iz+iJ)^2)}. \tag{159}$$

After differentiating with respect to J and performing the Grassmann integrations, the resolvent is given by

$$G(z) = \frac{1}{2\pi} \int dadb(z - ib)(\frac{(ib)^N}{a^N} + \frac{(ib)^{N-1}}{a^{N+1}})e^{-\frac{N}{2}((a-z)^2 + (b+iz)^2)}. \tag{160}$$

The integrals factorize into known integrals. In particular

$$\int_{-\infty}^{\infty} dx(ix)^n e^{-(x+iz)^2} = \frac{\sqrt{\pi}}{2^n}H_n(z). \tag{161}$$

We thus find that the result of the b integral is real. The integral over a is singular. Its calculation was discussed in a paper by Guhr [57]. The first observation is that a contains an infinitesimal increment $i\varepsilon$. By successive partial integrations we obtain

$$\int da\frac{1}{a^n}e^{-\frac{N}{2}(a-z)^2} = \frac{1}{(n-1)!}\int da\frac{1}{a}\partial_a^{n-1}e^{-\frac{N}{2}(a-z)^2}. \tag{162}$$

Using the definition of the Hermite polynomials

$$\partial_a^{n-1}e^{-\frac{N}{2}(a-z)^2} = \left(\frac{N}{2}\right)^{(n-1)/2}e^{-\frac{N}{2}(z-a)^2}H_{n-1}((z-a)\sqrt{N/2}) \tag{163}$$

and the relation

$$\text{Im}\frac{1}{a - i\varepsilon} = \pi\delta(a), \tag{164}$$

we find that

$$\text{Im}\int da\frac{1}{a^n}e^{-\frac{N}{2}(a-z)^2} = \frac{\pi}{(n-1)!}\left(\frac{N}{2}\right)^{(n-1)/2}e^{-\frac{N}{2}z^2}H_{n-1}(z\sqrt{N/2}). \tag{165}$$

To obtain the spectral density we use that

$$\rho(\lambda) = \frac{iN}{2\pi}(G(\lambda + i\varepsilon) - G(\lambda - i\varepsilon)). \tag{166}$$

Exercise. Use these results to derive an exact result for the spectral density in terms of Hermite polynomials.

7. THE SUPERSYMMETRIC METHOD OF RANDOM MATRIX THEORY: THE TWO-POINT FUNCTION

In this chapter we discuss the two-point correlation function for the GUE using the supersymmetric method. As a warm-up exercise we first consider the replica trick with fermionic replicas and bosonic replicas. Contrary to the one-point function we will obtain the effective partition function by using symmetry arguments. Both the replica calculations and the supersymmetric calculation can be found in [58]. Other useful sources are the book by Efetov [6], the papers by Zirnbauer [54, 35], the detailed review by Zuk [59] and the lecture notes by Fyodorov [34].

7.1. The Replica Trick

Before embarking on the supersymmetric method, we first study the calculation of the two-point function by means of the replica trick. In this case we have two possibilities for the generating function: a fermionic generating function or a bosonic generating function. The fermionic generating function for the two-point function is defined by

$$Z_n(J_1, J_2) = \int dH P(H) \det^n(x + J_1 + i\varepsilon + H) \det^n(y + J_2 - i\varepsilon + H), \qquad (167)$$

so that

$$\langle G(x + i\varepsilon)G(y - i\varepsilon)\rangle = \lim_{n \to 0} \frac{1}{n^2} \partial_{J_1}|_{J_1=0} \partial_{J_2}|_{J_2=0} \log Z_n(J_1, J_2) \qquad (168)$$

and can be represented as a Grassmann integral.

The bosonic generating function is defined by

$$Z_{-n}(J_1, J_2) = \int dH P(H) \det^{-n}(x + J_1 + i\varepsilon + H) \det^{-n}(y + J_2 - i\varepsilon + H) \qquad (169)$$

and can be represented as a bosonic integral. The two-point function is also given by (168).

7.1.1. The Fermionic Partition Function

The determinants in the fermionic partition function can be written as

$$\det^n(x + J_1 + i\varepsilon + H) \det^n(y + J_2 - i\varepsilon + H) = \int d\chi d\bar{\chi} \exp[-\bar{\chi}_\mu^k D_{\mu\nu} \chi_\nu^k], \qquad (170)$$

where $\chi_k = (\chi_1^1, \cdots, \chi_n^1, \chi_1^2, \cdots, \chi_n^2)$ and D is the block matrix

$$D = \begin{pmatrix} x + J_1 + i\varepsilon + H & 0 \\ 0 & y + J_2 - i\varepsilon + H \end{pmatrix} \equiv M + \begin{pmatrix} i\varepsilon + H & 0 \\ 0 & -i\varepsilon + H \end{pmatrix} \qquad (171)$$

For $x = y$, $J_1 = J_2$ and $\varepsilon = 0$ the symmetry of the generating function is $U(2n)$. One easily verifies that the resolvent changes sign when ε crosses the real axis so that for $x = y$ the expectation value

$$\langle \chi_1^{k*} \chi_1^k \rangle = -\langle \chi_2^{k*} \chi_2^k \rangle. \qquad (172)$$

This implies that the $U(2n)$ symmetry is broken spontaneously to $U(n) \times U(n)$. In the large N limit the massive modes can be integrated out and we are left with a theory of Goldstone modes. The Goldstone fields are in the coset $Q \in U(2n)/U(n) \times U(n)$ and are coupled to the microscopic variables as $\bar{\chi}_k Q_{kl} \chi_l$. Therefore, they transform as

$$Q \to UQU^{-1} \qquad (173)$$

under a transformation of the microscopic theory as

$$\chi_k \to U_{kl}\chi_l \quad \text{with} \quad U \in U(2n). \tag{174}$$

The $U(2n)$ invariance is broken by the mass term. However, invariance can be recovered if we transform the mass matrix according to

$$M \to UMU^{-1}. \tag{175}$$

The low energy-effective theory should have the same transformation properties. To first order in M we can write down only one invariant term,

$$\text{Tr}MQ, \tag{176}$$

resulting in the effective partition function

$$Z_n = \int_{Q \in U(2n)/U(n) \times U(n)} dQ \, e^{iN\Sigma \text{Tr}MQ}. \tag{177}$$

Here, dQ is the $U(2n)$ invariant measure to be discussed below. We will consider only the case that x and y are near the center of the spectrum. In particular, we take $x = \omega/2$ and $y = -\omega/2$ so that the mass matrix for $J_1 = J_2 = 0$ is equal to

$$M = \frac{\omega}{2}\Sigma_3 \quad \text{with} \quad \Sigma_3 = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & -\mathbf{1}_n \end{pmatrix}. \tag{178}$$

The constant Σ is determined by the requirement

$$\lim_{n \to 0} \partial_{J_1}|_{J_1=J_2=0} \frac{1}{nN} \log Z_n = \langle G(x) \rangle, \tag{179}$$

which is equal to $\pi\rho(0)/N$ for $x = 0$. The partition function (177) is the exact low energy limit of (167) in the thermodynamic limit for fixed ωN.

We can parameterize Q as,

$$Q = U\Sigma_3 U^{-1} \quad \text{with} \quad U = \exp\left[i\begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix}\right], \tag{180}$$

where A is an arbitrary complex matrix. In a polar decomposition of $A = v_1^{-1}\theta v_2$ with $v_1 \in U(n)$ and $v_2 \in U(n)/U^n(1)$ and θ a positive definite diagonal matrix, the matrix U is given by

$$U = \begin{pmatrix} v_1^{-1} & 0 \\ 0 & v_2^{-1} \end{pmatrix}\begin{pmatrix} \lambda & i\mu \\ i\mu & \lambda \end{pmatrix}\begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} \quad \text{with} \quad \mu = \sin\theta, \lambda = \cos\theta. \tag{181}$$

Therefore, Q can be written as

$$Q = \begin{pmatrix} v_1^{-1} & 0 \\ 0 & v_2^{-1} \end{pmatrix}\begin{pmatrix} \lambda^2 - \mu^2 & 2i\mu\lambda \\ -2i\mu\lambda & \lambda^2 - \mu^2 \end{pmatrix}\begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} \tag{182}$$

The integration measure in (177) is given by

$$\prod_{k,l}[\delta U_{12}]_{kl}[\delta U_{21}]_{kl}, \quad \text{with} \quad \delta U = U^{-1}dU. \tag{183}$$

The Jacobian from $[\delta U_{12}]_{kl}$ and $[\delta U_{21}]_{kl}$ to the variables $[\delta v_1]_{kl}$ and $[\delta v_2]_{kl}$ is given by the determinant (for $k \neq l$)

$$\det \begin{pmatrix} -i\lambda_k\mu_l & i\mu_k\lambda_l \\ i\mu_k\lambda_l & -i\lambda_k\mu_l \end{pmatrix} = \lambda_l^2 - \lambda_k^2. \tag{184}$$

For $k = l$ the transformation from $[\delta_{12}]_{kk}$ and $[\delta_{21}]_{kk}$ to the new variables $[\delta v_1]_{kk}$ and λ_k so that the Jacobian is given by

$$\det \begin{pmatrix} -i\mu_k\lambda_k & -i/\mu_k \\ i\mu_k\lambda_k & -i/\mu_k \end{pmatrix} = 2\lambda_k. \tag{185}$$

The integration measure for the new variables is therefore given by

$$\prod_{k,l}[\delta U_{12}]_{kl}[\delta U_{21}]_{kl} = \prod_k 2\lambda_k \prod_{k<l}(\lambda_k^2 - \lambda_l^2)^2. \tag{186}$$

Using $u_k \equiv \lambda_k^2 - \mu_k^2$ as new integration variables we finally obtain

$$Z_n(x) = \int_{-1}^{1}\prod_k du_k \prod_{k<l}(u_k - u_l)^2 e^{iN\omega\Sigma\Sigma_k u_k}, \tag{187}$$

with the thermodynamic limit taken at fixed

$$x = N\omega\Sigma. \tag{188}$$

The partition (187) can be written as a τ function. The first step is to expand the Vandermonde determinant

$$Z_n(x) = \int_{-1}^{1}\prod_k du_k \sum_{\sigma\pi} \text{sg}(\sigma\pi) u_1^{\sigma(1)+\pi(1)} \cdots u_n^{\sigma(n)+\pi(n)} e^{ix\Sigma_k u_k}, \tag{189}$$

where σ and π are permutations of $\{1,\cdots,n\}$, and sg denotes the sign of the permutation. Next we use that

$$\int_{-1}^{1}\prod_k du_k u_k^a e^{ixu_k} = (\partial_{ix})^a Z_1(x), \tag{190}$$

which results in

$$Z_n(x) = n![\det(\partial_{ix})^{i+j}Z_1(x)]_{0\leq i,j\leq n-1}. \tag{191}$$

7.1.2. The Bosonic Partition Function

The treatment of the bosonic partition function is quite similar to the fermionic partition function. Now the determinants in the partition function can be written as

$$\det{}^{-n}(x+J_1+i\varepsilon+H)\det{}^{-n}(y+J_2-i\varepsilon+H) = \frac{1}{(2\pi)^{2Nn}}\int d\phi_1 d\phi_2 \exp[i\phi_\mu^{*k}D_{\mu\nu}\phi_\nu^k],$$

(192)

where $\phi_k = (\phi_1^1,\cdots,\phi_1^n,\phi_2^1,\cdots,\phi_2^n)$ and D is the block matrix

$$D = \begin{pmatrix} x+J_1+i\varepsilon+H & 0 \\ 0 & -y-J_2+i\varepsilon-H \end{pmatrix} \equiv M + \begin{pmatrix} i\varepsilon+H & 0 \\ 0 & i\varepsilon-H \end{pmatrix}.$$

(193)

The essential difference with the fermionic case is that we have to make sure that the integrals are convergent. This is the reason that the sign of the lower right block of D has been reversed. For $x = y$, $J_1 = J_2$ and $\varepsilon = 0$ the symmetry of the generating function is not $U(2n)$ but rather $U(n,n)$. As in the fermionic case, this symmetry is broken spontaneously to $U(n) \times U(n)$. In this case, we have that for $x = y$

$$\langle \sum_k \phi_1^{k*}\phi_1^k \rangle = +\langle \sum_k \phi_2^{k*}\phi_2^k \rangle.$$

(194)

In the large N limit the massive modes can be integrated out by a leading order saddle point approximation and we are left with a theory of Goldstone modes. The Goldstone fields are in the coset $Q \in U(n,n)/U(n) \times U(n)$ and transform as

$$Q \to UQU^{-1}$$

(195)

under a transformation of the microscopic theory as

$$\phi_k \to U_{kl}\phi_l,$$

(196)

where $U \in U(n,n)$ satisfies

$$U^\dagger \Sigma_3 U = \Sigma_3.$$

(197)

In this case the Goldstone modes couple to the microscopic variables as $\phi_k^*\Sigma_3 Q_{kl}\phi_l$. The $U(n,n)$ invariance of the microscopic theory is broken by the mass term. However, this invariance is recovered if we transform the mass matrix by

$$M \to UMU^{-1}.$$

(198)

The low energy-effective theory should have the same transformation properties. To first order in M we can only write down the term

$$\text{Tr}MQ,$$

(199)

resulting in the effective partition function

$$Z_{-n} = \int_{Q \in U(n,n)/U(n) \times U(n)} dQ e^{iN\Sigma \text{Tr} MQ}.$$ (200)

The mass matrix is defined as in the fermionic case and the constant Σ is also determined by the requirement

$$\lim_{n \to 0} -\frac{1}{Nn} \partial_{J_1}|_{J_1=J_2=0} \log Z_{-n}(x=0) = \langle G(x=0)\rangle = \pi \rho(0)/N.$$ (201)

In the second identity we have used that $\text{Im}(x) > 0$. We can parameterize Q as

$$Q = U\Sigma_3 U^{-1} \quad \text{with} \quad U = \exp \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix},$$ (202)

where A is an arbitrary complex matrix. Notice that there is no factor i in the exponent in the definition of U. In a polar decomposition of $A = v_1^{-1}\theta v_2$ with $v_1 \in U(n)$ and $v_2 \in U(n)/U^n(1)$ and μ a positive definite diagonal matrix, the matrix U is given by

$$U = \begin{pmatrix} v_1^{-1} & 0 \\ 0 & v_2^{-1} \end{pmatrix} \begin{pmatrix} \lambda & \mu \\ \mu & \lambda \end{pmatrix} \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} \quad \text{with} \quad \mu = \sinh\theta \quad \text{and} \quad \lambda = \cosh\theta.$$ (203)

Therefore, Q can be written as

$$Q = \begin{pmatrix} v_1^{-1} & 0 \\ 0 & v_2^{-1} \end{pmatrix} \begin{pmatrix} \lambda^2 + \mu^2 & 2\mu\lambda \\ -2\mu\lambda & -\lambda^2 - \mu^2 \end{pmatrix} \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}.$$ (204)

Also in this case the integration measure is given by (186). The Jacobian from $[\delta U_{12}]_{kl}$ and $[\delta U_{21}]_{kl}$ to the variables $[\delta v_1]_{kl}$ and $[\delta v_1]_{kl}$ is given by the determinant (for $k \neq l$)

$$\det \begin{pmatrix} -\lambda_k \mu_l & \mu_k \lambda_l \\ \mu_k \lambda_l & -\lambda_k \mu_l \end{pmatrix} = \lambda_l^2 - \lambda_k^2.$$ (205)

For $k = l$ the new variables are $[\delta v_1]_{kk}$ and λ_k so that the Jacobian is given by

$$\det \begin{pmatrix} \mu_k \lambda_k & 1/\mu_k \\ -\mu_k \lambda_k & 1/\mu_k \end{pmatrix} = 2\lambda_k.$$ (206)

The integration measure is therefore also given by (186).

Using $u_k \equiv \lambda_k^2 + \mu_k^2$ as new integration variables we finally obtain

$$Z_{-n}(x) = \int_1^\infty \prod_k du_k \prod_{k<l} (u_k - u_l)^2 e^{ix\Sigma_k u_k}.$$ (207)

The bosonic partition function can also be written as a τ-function. By expanding the Vandermonde determinant we can express this generating function as a determinant of derivatives

$$Z_{-n}(x) = n! [\det(\partial_{ix})^{i+j} Z_{-1}]_{0 \leq i,j \leq n-1}.$$ (208)

7.2. The Supersymmetric Generating Function

Next we discuss the supersymmetric generation function for the two-point function defined by

$$Z(J_1, J_2) = \int dH P(H) \frac{\det(x + J_1 + i\varepsilon + H)}{\det(x - J_1 + i\varepsilon + H)} \frac{\det(y + J_2 - i\varepsilon + H)}{\det(y - J_2 - i\varepsilon + H)}. \tag{209}$$

Here, ε is an infinitesimal increment, and $P(H)$ is the Gaussian probability distribution for the unitary ensemble. The correlation function of two resolvents is given by

$$\langle G(x + i\varepsilon) G(y - i\varepsilon) \rangle = \frac{1}{4N^2} \partial_{J_1}|_{J_1 = 0} \partial_{J_2}|_{J_2 = 0} Z(J_1, J_2). \tag{210}$$

The generating function can be written in terms of Gaussian integrals

$$\begin{aligned} Z(J_1, J_2) &= \int dH P(H) \int d\psi d\psi^* e^{i\phi_1^*(x + J_1 + i\varepsilon + H)\phi_1 + i\chi_1^*(x - J_1 + i\varepsilon + H)\chi_1} \\ &\times e^{-i\phi_2^*(y + J_2 - i\varepsilon + H)\phi_2 + i\chi_2^*(y - J_2 - i\varepsilon + H)\chi_2}, \end{aligned} \tag{211}$$

where the vector ψ is defined by

$$\psi = \begin{pmatrix} \phi_1 \\ \chi_1 \\ \phi_2 \\ \chi_2 \end{pmatrix}, \tag{212}$$

and

$$d\psi = \prod_{\mu=1}^{N} \frac{d\chi_{1\mu} d\chi_{2\mu} d\phi_{1\mu} d\phi_{2\mu}}{\pi}. \tag{213}$$

The matrix D is defined by

$$\begin{aligned} D &= \begin{pmatrix} x + J_1 + i\varepsilon + H & 0 & 0 & 0 \\ 0 & x - J_1 + i\varepsilon + H & 0 & 0 \\ 0 & 0 & -[y + J_2 - i\varepsilon + H] & 0 \\ 0 & 0 & 0 & y - J_2 - i\varepsilon + H \end{pmatrix} \\ &\equiv LM + \begin{pmatrix} H & 0 & 0 & 0 \\ 0 & H & 0 & 0 \\ 0 & 0 & -H & 0 \\ 0 & 0 & 0 & H \end{pmatrix} \end{aligned} \tag{214}$$

Two widely used matrices that enter in the generating function are the matrix L, which breaks the symmetry between the 1-space and the 2-space, and the matrix k which breaks the supersymmetry. They are defined by

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv \begin{pmatrix} k_2 & 0 \\ 0 & k_2 \end{pmatrix}. \tag{215}$$

As in the previous sections we introduce the energy difference

$$\omega = x - y. \tag{216}$$

For $\omega = 0$ and $J_1 = J_2 = 0$ the partition function (209) is invariant under the transformations

$$\psi \to U\psi, \qquad \psi^* \to \psi^* U^\dagger \tag{217}$$

with

$$U^\dagger L U = L. \tag{218}$$

This group of superunitary transformations is known as $U(1,1|2)$. This symmetry is broken explicitly by the mass term $\psi^\dagger M \psi$ and spontaneously by the expectation value of

$$\langle \psi_i^* L_{ii} \psi_i \rangle = N\Sigma T_{3ii}. \tag{219}$$

Here, T_3 is defined by

$$T_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{220}$$

This result follows from the same analysis as in the case of bosonic or fermionic replicas. The symmetry of the partition function is thus broken spontaneously to $U(1|1) \times U(1|1)$. The Goldstone manifold is given $U(1,1/2)/U(1|1) \times U(1|1)$ and can be parameterized as

$$Q = U T_3 U^{-1}, \tag{221}$$

with $U \in U(1,1|2)$. The symmetry transformation (217) becomes a symmetry of the microscopic partition function if we also transform the mass term by

$$M \to U M U^{-1}. \tag{222}$$

Example. The mass term is given by $\psi^* L M \psi$. Under $\psi \to U \psi$ it transforms as

$$\psi^* U^\dagger L M U \psi = \psi^* U^\dagger L U U^{-1} M U \psi. \tag{223}$$

Since $U^\dagger L U = L$ we recover invariance if $M \to U M U^{-1}$.

Because the microscopic fields and the Goldstone fields are coupled as $\psi^* L Q \psi$, the Goldstone fields transforms as

$$Q \to U Q U^{-1}. \tag{224}$$

To lowest order in M we thus can write down only one nontrivial term with the same transformation properties and the microscopic generating function:

$$\text{Trg}MQ. \tag{225}$$

Notice that $Q^2 = 1$.

As before we will study the two-point function in the center of the spectrum. For zero sources the mass matrix is given by

$$M = \frac{\omega}{2}T_3. \tag{226}$$

We are interested in the thermodynamic limit at fixed ωN. To leading order in $1/N$ the partition function is given by the integral over the saddle-point manifold,

$$Z(J_1, J_2) = \int_{Q \in U(1,1|2)/U(1|1) \times U(1|1)} dQ e^{i\frac{\omega N}{2}\Sigma\text{Trg}(T_3 Q) + J_1 iN\Sigma\text{Trg}k_2 Q_{11} + J_2 iN\Sigma\text{Trg}k_2 Q_{22}}, \tag{227}$$

with the Goldstone manifold can be parameterized as as in (221). The supersymmetry breaking matrix k_2 was introduced in (215) and is given by by

$$k_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{228}$$

The subscripts 11 and 22 refer to the projection onto the respective blocks of the Q matrices. By differentiation with respect to the source terms we thus obtain

$$\langle G(x+i\varepsilon)G(y-i\varepsilon)\rangle = -\frac{\Sigma^2}{4}\int_{Q \in U(1,1|2)/U(1|1) \times U(1|1)} dQ\text{Trg}(kQ_{11})\text{Trg}(kQ_{22})e^{\frac{i\omega N}{2}\Sigma\text{Trg}(T_3 Q)}. \tag{229}$$

We have seen that the boson-boson block of Q is noncompact. This results in a mass with a positive real part when Q is expanded in the generators of U. Because of the supertrace we have an additional minus sign for the fermion-fermion term which would result in a negative real part. To compensate this minus sign we have to include an extra factor i in the generators of U so that the fermion-fermion block becomes compact. The integral is therefore over the maximum Riemannian submanifold of $Gl(2|2)/Gl(1|1) \times Gl(1|1)$ and the measure is the invariant measure of the unbroken group [44, 60].

If x and y would be on the same side of the real axis, the matrix T_3 would not appear in the partition function, and the symmetry between the 1-variable and the 2-variables would not be spontaneously broken. In that case there is no Goldstone manifold and, to leading order in $1/N$ the integrals over the 1-variables and 2-variables factorize. Therefore, they only contribute to the disconnected piece of the correlation function.

7.3. The Integration Manifold

In order to identify the integration manifold, we will be guided by the principle of convergence. Grassmann integrals are always convergent, and we will ignore them for

the moment. Then fermion-fermion blocks and the boson-boson blocks decouple and can be treated separately. This corresponds to the case of one fermionic replica or one bosonic replica discussed earlier in this chapter.

In the case of one bosonic replica for x and one for y, the symmetry transformation satisfies

$$U^\dagger \sigma_3 U = \sigma_3 \tag{230}$$

with σ_3 the third Pauli matrix. If we parameterize U as

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{231}$$

we obtain the equations

$$
\begin{aligned}
a^*a - b^*b &= 1, \\
c^*c - d^*d &= -1, \\
ac^* &= bd^*, \\
ca^* &= db^*.
\end{aligned}
\tag{232}
$$

Combining these equations we find that $|a| = |d|$ and $|b| = |c|$. In our partition function, the matrix U only enters in the combination of the coset $U\Sigma_3 U^{-1}$. Without loss of generality we can omit the phase factors of a and d. Then we obtain $b = c^*$. Our result for the parameterization of U is thus given by

$$U = \begin{pmatrix} \sqrt{1 + a^*a} & a \\ a^* & \sqrt{1 + a^*a} \end{pmatrix}. \tag{233}$$

Next we consider the fermion-fermion block which enters in the effective partition function as

$$e^{i\frac{N\omega}{2}[\mathrm{Tr}Q^{BB} - \mathrm{Tr}Q^{FF}]}. \tag{234}$$

With some foresight we have chosen the signs of the Hamiltonian the same in the two fermionic integrals in the generating function. The symmetry of the fermionic variables is therefore $U(2)$. The Goldstone manifold is still given by $Q_{FF} = U\sigma_3 U^{-1}$. Because U is now compact the expansion of Q_{FF} to second order in the generators of U results in a convergent integral. If we would have started with a noncompact formulation also for the fermionic variables the integral over Q_{FF} would not have been convergent because of the additional minus sign due to the supertrace. In fact, we would have been forced to rotate the integration contour of Q_{FF} so that we again would have arrived at a compact parameterization of Q_{FF}.

Convergence of the integrals thus forces us to parameterize U according to the compact subgroup of $Gl(2)$ which is $U(2)$. Therefore, $Q_{FF} \in U(2)/U(1) \times U(1)$, which can be parameterized as

$$\begin{pmatrix} \cos\theta & ie^{i\phi}\sin\theta \\ ie^{-i\phi}\sin\theta & \cos\theta \end{pmatrix}. \tag{235}$$

More suggestively this can be written as

$$U = \begin{pmatrix} \sqrt{1-b^*b} & ib \\ ib^* & \sqrt{1-b^*b} \end{pmatrix}, \tag{236}$$

We observe that the parameterization of the fermion-fermion variable is structurally the same as the parameterization of the boson-boson variables.

Next we consider the Grassmann variables. A matrix $U \in U(1,1|2)$ contains 4 Grassmannian entries. However, in the coset U four of them can be absorbed by the $U(1|1) \times U(1|1)$ subgroup. Our final result for the parameterization of U is then given by

$$U = \begin{pmatrix} \sqrt{1+t_{12}t_{21}} & t_{12} \\ t_{21} & \sqrt{1+t_{21}t_{12}} \end{pmatrix}, \tag{237}$$

with

$$t_{12} = \begin{pmatrix} a & \alpha \\ \beta & ib \end{pmatrix}, \qquad t_{21} = \begin{pmatrix} a^* & \gamma \\ \delta & ib^* \end{pmatrix}, \tag{238}$$

and α, β, γ and δ Grassmann variables. From (218) we find a relation between the Grassmann variables by conjugation. However, because a Grassmann variable and its conjugate are independent integration variables, the conjugation relation can be ignored as has been done in the above parameterization.

7.4. The Invariant Measure

Now that we have found a parameterization of the integration manifold we are able to calculate the invariant measure. From measure theory we know that the invariant measure is given by

$$\mathrm{detg} \left(\frac{\delta U_{12} \delta U_{21}}{\delta t_{12} \delta t_{21}} \right) dt_{12} dt_{21} \tag{239}$$

with $\delta U = U^{-1} dU$. This measure is invariant under $U(1,1|2)$ and left invariant under $U(1|1) \times U(1|1)$ [60].

Instead of calculating the graded determinant for the general case we hope to convince the reader that it is equal to unity by evaluating it for two special cases. The general case is left as an exercise.

Example. We consider the bosonic matrix

$$U_1 = \begin{pmatrix} \sqrt{1+a^*a} & a \\ a^* & \sqrt{1+a^*a} \end{pmatrix}. \tag{240}$$

In order to calculate the invariant measure on the space of U matrices we have to evaluate

$$U_1^{-1} dU_1 = \begin{pmatrix} \frac{1}{2}(a^* da - a da^*) & \sqrt{1+a^*a}\, a da - \frac{a(a da^* + a^* da)}{2\sqrt{1+a^*a}} \\ \sqrt{1+a^*a}\, a da^* - \frac{a^*(a da^* + a^* da)}{2\sqrt{1+a^*a}} & \frac{1}{2}(a da^* - a^* da) \end{pmatrix}.$$

The Jacobian for the transformation from the variables $(U_1^{-1}dU_1)_{12}$ and $(U_1^{-1}dU_1)_{21}$ to the variables a and a^* is given by the determinant of the matrix

$$\begin{pmatrix} \sqrt{1+a^*a} - \frac{a^*a}{2\sqrt{1+a^*a}} & -\frac{a^2}{2\sqrt{1+a^*a}} \\ -\frac{a^{*2}}{2\sqrt{1+a^*a}} & \sqrt{1+a^*a} - \frac{a^*a}{2\sqrt{1+a^*a}} \end{pmatrix}, \tag{242}$$

which is easily seen to be equal to unity. The invariant measure is thus simply given by $da\,da^*$.

Example. Next we consider a more complicated example with U_2 given by

$$U_2 = \begin{pmatrix} 1+\frac{\alpha\delta}{2} & 0 & 0 & \alpha \\ 0 & 1+\frac{\beta\gamma}{2} & \beta & 0 \\ 0 & \gamma & 1+\frac{\gamma\beta}{2} & 0 \\ \delta & 0 & 0 & 1+\frac{\delta\alpha}{2} \end{pmatrix}. \tag{243}$$

For $U_2^{-1}dU_2$ we obtain

$$\begin{pmatrix} -\frac{1}{2}(\alpha d\delta + \delta d\alpha) & 0 & 0 & d\alpha \\ 0 & -\frac{1}{2}(\gamma d\beta + \beta d\gamma) & d\beta & 0 \\ 0 & d\gamma & -\frac{1}{2}(\gamma d\beta + \beta d\gamma) & 0 \\ d\delta & 0 & 0 & -\frac{1}{2}(\alpha d\delta + \delta d\alpha) \end{pmatrix}. \tag{244}$$

One easily sees that the Jacobian for the transformation from the 12 and 21 blocks of $U_2^{-1}dU_2$ to α, β, γ and δ is equal to unity. The invariant measure is therefore given by $d\alpha\,d\beta\,d\gamma\,d\delta$.

Exercise. Calculate the Berezinian for the general case and show that its value is equal to unity as well . The invariant measure is thus given by $dt_{12}dt_{21}$.

In terms of the t_{12} variables the measure is flat with invariant line element given by

$$ds^2 = \mathrm{Tr}g\,dt_{12}dt_{21}. \tag{245}$$

This measure is invariant under the transformations

$$t_{12} \to ut_{12}v^{-1}, \qquad t_{21} \to vt_{21}u^{-1}. \tag{246}$$

7.5. Transformation to Polar Variables.

Although an explicit calculation of the Grassmann integrations using the parameterization of the previous section is possible, the algebra of such calculation is quite

laborious. For that reason we introduce new integration variables defined by the polar decomposition

$$t_{12} = u \begin{pmatrix} \mu_1 & 0 \\ 0 & i\mu_2 \end{pmatrix} v^{-1}, \qquad (247)$$

and

$$t_{21} = v \begin{pmatrix} \mu_1^* & 0 \\ 0 & i\mu_2^* \end{pmatrix} u^{-1}, \qquad (248)$$

with u and v given by

$$u = \begin{pmatrix} 1 + \frac{\alpha\beta}{2} & \alpha \\ \beta & 1 - \frac{\alpha\beta}{2} \end{pmatrix}, \qquad v = \begin{pmatrix} 1 + \frac{\gamma\delta}{2} & \gamma \\ \delta & 1 - \frac{\gamma\delta}{2} \end{pmatrix}. \qquad (249)$$

The matrices u and v satisfy the property that

$$u\sigma_3 u = \sigma_3, \qquad v\sigma_3 v = \sigma_3, \qquad (250)$$

with σ_3 the third Pauli-matrix. Notice that we have not imposed a conjugation on the Grassmann variables. This means that for the Grassmann variables we have relaxed the condition [44] $U^\dagger L U = L$ that defines $U(1,1|2)$.

To calculate the Berezinian of the polar variables we consider the variations

$$\begin{aligned} u^{-1} dt_{12} v &= u^{-1} du\mu + d\mu - \mu v^{-1} dv, \\ v^{-1} dt_{21} u &= v^{-1} dv\mu^* + d\mu^* - \mu^* u^{-1} du, \end{aligned} \qquad (251)$$

where μ and μ^* are diagonal matrices with diagonal elements μ_1 and $i\mu_2$, and μ_1^* and $i\mu_2^*$, respectively. The block of the Berezinian corresponding to differentiation of the boson-fermion blocks of $u^{-1}\delta t_{12} v$ and $v^{-1}\delta t_{21} u$ with respect to μ and μ^* is equal to zero. The determinant of the block corresponding to differentiation of the boson-boson block with respect to these variables is equal to one. The only nontrivial contribution to the Berezinian then arises from the variation of the boson-fermion blocks of $u^{-1} dt_{12}$ and $v^{-1} dt_{21} u$ with respect to the off-diagonal elements of $u^{-1} du$ and $v^{-1} dv$. The Berezinian from the t_{12} and t_{21} variables to the polar variables is thus given by the inverse determinant of

$$\begin{pmatrix} i\mu_2 & 0 & -\mu_1 & 0 \\ 0 & \mu_1 & 0 & -i\mu_2 \\ -\mu_1^* & 0 & i\mu_2^* & 0 \\ 0 & -i\mu_2^* & 0 & \mu_1^* \end{pmatrix}. \qquad (252)$$

The value of the Berezinian is thus equal to

$$B = \frac{1}{(\mu_1^* \mu_1 + \mu_2^* \mu_2)^2}. \qquad (253)$$

To derive the final result for the two-point function we also need

$$\begin{aligned}
\mathrm{Tr} g_k 2 Q_{11} &= \lambda_1 + \lambda_2 + 2\alpha\beta(\lambda_1 - \lambda_2), \\
\mathrm{Tr} g_k 2 Q_{22} &= -\lambda_1 - \lambda_2 - 2\gamma\delta(\lambda_1 - \lambda_2),
\end{aligned} \tag{254}$$

where λ_1 and λ_2 are the diagonal elements of Q_{11} given by

$$\begin{aligned}
\lambda_1 &= 1 + 2|\mu_1|^2, \\
\lambda_2 &= 1 - 2|\mu_2|^2.
\end{aligned} \tag{255}$$

For the exponent in the partition function we obtain

$$e^{iN\omega\Sigma(\lambda_1 - \lambda_2)}. \tag{256}$$

After performing the angular integrations, the final result for the two-point correlation function is given by

$$\begin{aligned}
\langle G(x+i\varepsilon)G(y-i\varepsilon)\rangle &= \frac{\Sigma^2}{4}\int_1^\infty d\lambda_1 \int_{-1}^1 d\lambda_2 \frac{d\alpha d\beta d\gamma d\delta}{(\lambda_1 - \lambda_2)^2} e^{iN\omega\Sigma(\lambda_1 - \lambda_2)} \\
&\quad \times (\lambda_1 + \lambda_2 + 2\alpha\beta(\lambda_1 - \lambda_2))(\lambda_1 + \lambda_2 + 2\gamma\delta(\lambda_1 - \lambda_2)).
\end{aligned} \tag{257}$$

A factor of $1/(2\pi)^2$ in the integration measure has been canceled by the angular integrations. According to the Efetov-Zirnbauer theorem, the terms of second order in the Grassmann variables do not contribute to the final result. Because of the extra factor $\lambda_1 - \lambda_2$ there are no problems with singularities. However, the term of zeroth order in the Grassmann variables is the product of a divergent result, from the integration over the bosonic variables, and zero from the Grassmann integrations. In this case, a careful treatment of the singularity will result in a finite contribution known as an Efetov-Wegner term.

7.6. Final Result and Efetov-Wegner Terms

We first consider the term of zeroth order in the Grassmann variables. In section (5.4) we discussed the appearance of Efetov-Wegner terms for a Gaussian integral. Following the same procedure for the present case, we will regularize the integrals by excluding a small region around the singular point $a = b = 0$. Using the procedure of ref. [56] we replace the integration measure by

$$da\,da^*\,db\,db^* \rightarrow da\,da^*\,db\,db^*\,\theta(\sqrt{a^*a + b^*b} - \rho), \tag{258}$$

and take the limit $\rho \rightarrow 0$ at the end of the calculation. We express this regularized measure in terms of the polar variables μ_1, μ_1^*, μ_2 and μ_2^*. Some algebra using the explicit polar decomposition of t_{12} and t_{21} results in

$$\begin{aligned}
a^*a + b^*b = \quad & \mu_1^*\mu_1 + \mu_2^*\mu_2 + (\mu_1^*\mu_1 - \mu_2^*\mu_2)(\alpha\beta + \gamma\delta) + 2i\mu_1\mu_2^*\beta\gamma + 2i\mu_1^*\mu_2\delta\alpha \\
& + \frac{3}{2}\alpha\beta\gamma\delta(\mu_1^*\mu_1 + \mu_2^*\mu_2),
\end{aligned} \tag{259}$$

For $\sqrt{a^*a + b^*b}$ we obtain

$$\sqrt{a^*a + b^*b} = R + \frac{1}{2R}[(\mu_1^*\mu_1 - \mu_2^*\mu_2)(\alpha\beta + \gamma\delta) + 2i\mu_1\mu_2^*\beta\gamma + 2i\mu_1^*\mu_2\delta\alpha] + \frac{R}{2}\alpha\beta\gamma\delta,$$

(260)

where

$$R = \sqrt{\mu_1^*\mu_1 + \mu_2^*\mu_2} = \sqrt{\lambda_1 - \lambda_2}.$$

(261)

Keeping only the terms of zeroth and maximum order in the Grassmann variables the θ function can be expanded as

$$\theta(\sqrt{a^*a + b^*b} - \rho) = \theta(R - \rho) + \frac{R}{2}\delta(R - \rho)\alpha\beta\gamma\delta + \frac{R^2}{4}\partial_R\delta(R - \rho)\alpha\beta\gamma\delta. \quad (262)$$

The derivative of the delta function can be converted into a delta function by partial integration. Since our contribution comes from the singularity we only have to take into account derivatives of the singular terms. Near $R \to 0$ the integrand containing the $\delta-$ functions is of the form

$$\frac{dR}{R}\left[\frac{R}{2}\delta(R - \rho) + \frac{R^2}{4}\partial_R\delta(R - \rho)\right]$$

(263)

times an angular factor. Partial integration of the second term results in

$$\frac{dR}{R}\left[\frac{R}{2}\delta(R - \rho) - \frac{R}{4}\delta(R - \rho)\right] = \frac{dR}{R}\frac{R}{4}\delta(R - \rho) = \frac{dR}{R}\frac{R^2}{2}\delta(R^2 - \rho^2).$$

(264)

Performing the Grassmann integration and including the Efetov-Wegner terms (264) after transforming back to the λ_1, λ_2 variables in the correlation function (257 results in

$$\langle G(x + i\varepsilon)G(y - i\varepsilon)\rangle = \frac{\Sigma^2}{4}\int_1^\infty d\lambda_1 \int_{-1}^1 d\lambda_2 \left[\frac{\delta(\lambda_1 - \lambda_2 - \rho^2)}{2(\lambda_1 - \lambda_2)}(\lambda_1 + \lambda_2)^2 + 4\right]e^{iN\omega\Sigma(\lambda_1 - \lambda_2)}.$$

(265)

The first integral can be performed as follows

$$\int_1^\infty d\lambda_1 \int_{-1}^1 d\lambda_2 \frac{2\delta(\lambda_1 - \lambda_2 - \rho^2)}{\lambda_1 - \lambda_2}(\lambda_1 + \lambda_2)^2 = \int_{1-\rho^2}^1 d\lambda_2 \frac{2(\rho^2 + 2\lambda_2)^2}{\rho^2} = 8,$$

(266)

where we have taken the limit $\rho \to 0$. The final result for the correlation function is given by

$$\langle G(x + i\varepsilon)G(y - i\varepsilon)\rangle = \Sigma^2 + \frac{\Sigma^2}{4}\int_1^\infty d\lambda_2 \int_{-1}^1 d\lambda_2 e^{iN\omega\Sigma(\lambda_1 - \lambda_2)} = \Sigma^2 + 2i\frac{\sin(N\omega\Sigma)e^{iN\omega\Sigma}}{(\omega N)^2},$$

(267)

which is the well-known result for the two-point correlation function of the Gaussian Unitary Ensemble. The first term is the disconnected piece of the correlation function.

8. INTEGRABILITY AND THE REPLICA TRICK

Integrability and Random Matrix Theory are closely related. In this section we discuss the Toda lattice equation and show that the two-point function of the GUE can be obtained from the replica limit of the Toda lattice equation [61]. A closely related method which preceded this work is the replica limit of the Painlevé equation. For a discussion of this approach we refer to the original literature [62].

8.1. The Two-Point Function of the Gaussian Unitary Ensemble

In this section we derive the two-point function of the Gaussian Unitary Ensemble from the Replica limit of the Toda lattice equation. In section 7.1.1 we have shown that the partition function with n fermionic flavors can written as [63]

$$Z_n(x) = C_n[\det(\partial_{ix})^{i+j}Z_1]_{0\leq i,j\leq n-1}, \tag{268}$$

and n bosonic flavors as [61]

$$Z_{-n}(x) = n![\det(\partial_{ix})^{i+j}Z_{-1}]_{0\leq i,j\leq n-1}, \tag{269}$$

where

$$Z_1(x) = \int_{-1}^{1} du e^{ixu} \qquad \text{and} \qquad Z_{-1}(x) = \int_{1}^{\infty} du e^{ixu}. \tag{270}$$

We will now show that the bosonic and fermionic partition functions are related by a recursion relation. The main ingredient of this derivation is the Sylvester identity [64, 27] which is valid for determinant of an arbitrary matrix A. It is given by

$$C_{ij}C_{pq} - C_{iq}C_{pj} = \det(A)C_{ij;pq}, \tag{271}$$

where the C_{ij} are cofactors of the matrix A and the $C_{ij;pq}$ are double cofactors. We apply this identity for $i = j = n-1$ and $i = j = n$, and remind the reader that differentiating a determinant is differentiating either with respect to all columns or with respect to all rows. Because of the derivative structure of the partition function, only differentiating the last column or the last row gives a nonzero result, but this is exactly the desired cofactor of the matrix with n increased by 1. One can easily show that the relevant cofactors are given by

$$
\begin{aligned}
C_{nn} &= Z_{n-1}(x), \\
C_{n-1n} &= -\partial_{ix}Z_{n-1}(x), \\
C_{nn-1} &= -\partial_{ix}Z_{n-1}(x), \\
C_{n-1n-1} &= (\partial_{ix})^2 Z_{n-1}(x), \\
C_{n-1n-1;nn} &= Z_{n-2}(x).
\end{aligned} \tag{272}
$$

By increasing the index by one we obtain the Toda lattice equation

$$\partial_{ix}^2 \log Z_n(x) = n^2 \frac{Z_{n+1}(x)Z_{n-1}(x)}{[Z_n(x)]^2} \tag{273}$$

where the factor n^2 follows from the choice of the normalization constants. We have made this choice because the left hand side is proportional to n^2. The replica limit of the two-point correlation is given by

$$
\begin{aligned}
R_2(x) &= -\lim_{n \to 0} \frac{1}{n^2} \partial_x^2 \log Z_n(x) = Z_1(x)Z_{-1}(x) \\
&= \int_{-1}^{1} du\, e^{iux} \int_{1}^{\infty} e^{iux} \\
&= 2i \frac{\sin x}{x} \frac{e^{ix}}{x},
\end{aligned} \tag{274}
$$

which agrees with the exact analytical result for the two-point function. The fermionic partition functions, the bosonic partition functions and the super-symmetric partition function form a single integrable hierarchy which are related by the Toda lattice equation [61]. A closely related way to derive the two-point function of the GUE is to take the replica limit and the corresponding Painlevé equation. This has the advantage that the bosonic partition function does not directly enter in the derivation. For a discussion of this result we refer to [62] which preceded our work [61] on the Toda lattice.

8.2. Toda Lattice Equation

The Toda lattice equation describes a one dimensional lattice in which neighboring atoms interact via a potential that depends exponentially on the distance. The Hamiltonian of such system is given by

$$H = \frac{1}{2} \sum_{k=-\infty}^{\infty} p_k^2 + \sum_{k=-\infty}^{\infty} e^{-(q_k - q_{k-1})}. \tag{275}$$

This integrable system has infinitely many constants of motion. For example, this can be concluded from the existence of a Lax pair meaning that the Hamiltonian equations of motion can be written in the form

$$\frac{dK}{dt} = [K, L], \tag{276}$$

where K and L are two operators. Then

$$\frac{d\mathrm{Tr}[K^n]}{dt} = n\mathrm{Tr}[K^{n-1}\frac{dK}{dt}] = n\mathrm{Tr}[K^{n-1}[K, L]] = 0, \tag{277}$$

by using the cyclic invariance of the trace.

The Hamilton equations of motion can be written as

$$\frac{d^2}{dt^2} q_k = e^{q_{k-1}-q_k} - e^{q_k-q_{k+1}}. \tag{278}$$

If we define

$$z_{k+1} = z_k e^{-q_k}, \tag{279}$$

we obtain the recursion relation

$$-\frac{d^2}{dt^2} \log z_{k+1} + \frac{d^2}{dt^2} \log z_k = \frac{z_{k+1} z_{k-1}}{z_k^2} - \frac{z_{k+2} z_k}{z_{k+1}^2}. \tag{280}$$

We then conclude that

$$\frac{d^2}{dt^2} \log z_k = \frac{z_{k+1} z_{k-1}}{z_k^2} + c_0. \tag{281}$$

The constant c_0 can be eliminated by a redefinition of $z_k \rightarrow z_k \exp(c_0 t^2/2)$.

This equation has been studied extensively in the literature and has appeared in different areas of physics. As an example, we mention applications to Chern-Simons theory [65] and other applications to Random Matrix Theory [66, 27]. The application of the Toda lattice equation discussed in the previous section [61] illustrates the power of integrability which makes it possible to obtain exact nonperturbative by means of the replica trick. More applications along these lines will be discussed later in these lectures. Other applications are parametric correlations and the QCD_3 spectral density [67] which both have been analyzed by the supersymmetric method [68, 69].

9. SYMMETRIES OF THE QCD PARTITION FUNCTION

It is well-known that the QCD action is greatly constrained by gauge symmetry, Poincaré invariance and renormalizability. These symmetries determine the structure of the Dirac operator. In this section we will discuss the global symmetries of the Euclidean Dirac operator. They play an essential role for its spectral properties in the deepest infrared. In particular, the chiral symmetry, the flavor symmetry and the anti-unitary symmetry of the continuum Dirac operator are discussed.

9.1. The QCD partition function

The QCD partition function in a box of volume $V_3 = L^3$ can be expressed in terms of the eigenvalues of the QCD Hamiltonian E_k as

$$Z^{QCD} = \sum_k e^{-\beta E_k}, \tag{282}$$

where β is the inverse temperature. At low temperatures ($\beta \to \infty$) the partition function is dominated by the lightest states of the theory, namely the vacuum state, with an energy density of E_0/V_3, and massless excitations thereof. The partition function Z_{QCD} can be rewritten as a Euclidean functional integral over the nonabelian gauge fields A_μ,

$$Z^{QCD}(M) = \int dA_\mu \prod_{f=1}^{N_f} \det(D + m_f) e^{-S_{YM}/g^2},\tag{283}$$

where S_{YM} is the Yang-Mills action given by

$$S_{YM} = \int d^4 [-\frac{1}{4} F_{\mu\nu}^{a}{}^2 + i\frac{\theta}{32\pi^2} F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a].\tag{284}$$

The field strength is given by

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{abc} A_\mu^b A_\nu^c.\tag{285}$$

The f_{abc} are the structure constants of the gauge group $SU(N_c)$. We also denote the gauge fields by

$$A_\mu = A_{\mu a} \frac{T^a}{2},\tag{286}$$

where T^a are the generators of the gauge group. The dual of the field strength is defined by

$$\tilde{F}_{\mu\nu} = \frac{1}{4} \varepsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}.\tag{287}$$

The integral $\nu \equiv \frac{1}{32\pi^2} \int d^4 x F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a$ is a topological invariant, i.e. it does not change under continuous transformations of the gauge field. An important class of field configurations are instantons. These are topological nontrivial field configurations that minimize the classical action which are classified according to their topological charge ν. The parameter θ in (284) is known as the θ-angle. Experimentally, its value is consistent with zero. In (283) the mass matrix is diagonal, $M = \text{diag}(m_1, \cdots, m_{N_f})$, but below we will also consider a general mass matrix.

The anti-Hermitian Dirac operator in (283) is defined by

$$D = \gamma_\mu (\partial_\mu + iA_\mu),\tag{288}$$

where the γ_μ are the Euclidean Dirac matrices with anti-commutation relation

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}.\tag{289}$$

In the chiral representation the γ-matrices are given by

$$\gamma_k = \begin{pmatrix} 0 & i\sigma_k \\ -i\sigma_k & 0 \end{pmatrix} \quad \gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.\tag{290}$$

In this representation the Dirac operator has the structure

$$D = \begin{pmatrix} 0 & id \\ id^\dagger & 0 \end{pmatrix}. \tag{291}$$

The integration measure can be defined by discretizing space time

$$dA_\mu^a = \prod_x dA_\mu^a(x). \tag{292}$$

A particular popular discretization is the lattice discretization where the QCD action is discretized on a hyper-cubic lattice with spacing a. The discussion of lattice QCD would be a lecture by itself. For the interested reader we recommended several excellent textbooks on the subject [70, 71, 72].

Alternatively, we may define the measure in terms of a complete set of orthonormal functions $\phi_{n\,\mu}^a$. In terms of the expansion

$$A_\mu^a = \sum_{n=1}^{N} a_n \phi_{n\,\mu}^a \tag{293}$$

the measure is defined as

$$\prod_n da_n. \tag{294}$$

A field theory is obtained by taking the continuum limit, i.e. the limit $a \to 0$ or $N \to \infty$ for the integration measures discussed above. This limit only exists if we simultaneously adjust the coupling constant, i.e. $g \to g(a)$ or $g(N)$. If such limit exists the field theory is called renormalizable. For QCD it turns out that this adjusted coupling constant approaches zero in the continuum limit, a property known as asymptotic freedom.

We will be mainly interested in the eigenvalues of the Euclidean Dirac operator and how they fluctuate for gauge fields A_μ distributed according to the QCD action. We will show that below a well-defined scale the fluctuations of the Dirac eigenvalues are given by the corresponding chiral RMT discussed in the first chapter. It turns out that QCD and chiral RMT have the same low energy limit which we will analyze by means of the supersymmetric method.

9.2. Axial Symmetry

The eigenvalues of the QCD Dirac operator are given by

$$D\phi_k = i\lambda_k \phi_k. \tag{295}$$

The eigenvalues are purely imaginary because the Dirac operator is anti-Hermitian (notice that $\partial_\mu^\dagger = -\partial_\mu$ and $A_\mu^\dagger = A_\mu$).

The axial symmetry, or the $U_A(1)$ symmetry, can be expressed as the anti-commutation relation

$$\{\gamma_5, D\} = 0. \tag{296}$$

This implies that all nonzero eigenvalues occur in pairs $\pm i\lambda_k$ with eigenfunctions given by ϕ_k and $\gamma_5\phi_k$. If $\lambda_k = 0$ the possibility exists that $\gamma_5\phi_k \sim \phi_k$, so that $\lambda_k = 0$ is an unpaired eigenvalue. According to the Atiyah-Singer theorem, the total number of such zero eigenvalues is a topological invariant, i.e., it does not change under continuous transformations of the gauge field configuration. Indeed, this possibility is realized by the field of an instanton which is a solution of the classical equations of motion. On the other hand, it cannot be excluded that $\lambda_k = 0$ while ϕ_k and $\gamma_5\phi_k$ are linearly independent. However, this imposes additional constraints on the gauge fields that will be violated by infinitesimal deformations. Generically, such situation does not occur.

In a decomposition according to the total number of topological zero modes, the QCD partition function can be written as

$$Z^{QCD}(M) = \sum_\nu e^{i\nu\theta} Z_\nu^{QCD}(M), \tag{297}$$

where

$$Z_\nu^{QCD}(M) = \langle \prod_f m_f^\nu \prod_k (\lambda_k^2 + m_f^2) \rangle_\nu. \tag{298}$$

Here, $\langle \cdots \rangle_\nu$ denotes the average over gauge-field configurations with topological charge ν weighted by the Yang-Mills action. If we introduce right-handed and left-handed masses as complex conjugated masses we find that the θ dependence of the QCD partition function is only through the combination $me^{i\theta/N_f}$. This property can be used to obtain the θ-dependence of the low energy effective partition function.

Exercise. Show that the θ dependence of the QCD partition function is only through the combination $e^{i\theta/N_f}$.

9.3. Flavor Symmetry

A second important global symmetry is the flavor symmetry. This symmetry can be best explained by writing the fermion determinant in the QCD partition function as a functional integral over Grassmann variables,

$$\prod_f \det(D + m_f) = \int d\psi \, d\bar\psi \, e^{\int d^4x \sum_{f=1}^{N_f} \bar\psi^f (D + m_f) \psi^f}. \tag{299}$$

In a chiral basis with $\psi_R = \gamma_5 \psi_R$ and $\psi_L = -\gamma_5 \psi_L$, the exponent can be rewritten as

$$\sum_{f=1}^{N_f} \bar\psi^f (D + m_f) \psi^f = \bar\psi_R^f D \psi_R^f + \bar\psi_L^f D \psi_L^f + \bar\psi_R^f M_{RL} \psi_L^f + \bar\psi_L^f M_{LR} \psi_R^f,$$

$$\tag{300}$$

where we have used that

$$\bar\psi_R^f D\psi_L^f = \bar\psi_R^f \gamma_5 D\gamma_5 \psi_L^f = -\bar\psi_R^f D\psi_L^f = 0, \tag{301}$$

and similarly for the matrix elements between a left-handed and a right-handed state. To better illuminate the transformation properties of the partition function we have replaced the diagonal mass matrix by M_{RL} and M_{LR}.

Exercise. Construct ψ_R, ψ_L, $\bar\psi_R$ and $\bar\psi_L$ from the original basis.

For $m_f = 0$ we have the symmetry

$$\begin{aligned} \psi_L &\rightarrow U\psi_L, \quad \bar\psi_L \rightarrow \bar\psi_L U^{-1}, \\ \psi_R &\rightarrow V\psi_R, \quad \bar\psi_R \rightarrow \bar\psi_R V^{-1}. \end{aligned} \tag{302}$$

The only condition to be imposed on U and V is that their inverse exists. If the number of left-handed modes is equal to the number of right-handed modes we thus have an invariance under $Gl_R(N_f) \times Gl_L(N_f)$, where $Gl(N_f)$ is the group of complex $N_f \times N_f$ matrices with nonzero determinant. However, if the number of left-handed modes is not equal to the number of right-handed modes, the axial-symmetry group is broken to an $Sl(N_f)$ subgroup whereas the vector symmetry with $U = V$ remains unbroken. For $m = 0$ the flavor symmetry is thus broken explicitly to $Gl_V(N_f) \times Sl_A(N_f)$ by instantons or the anomaly. A $Gl_V(1)$ subgroup of $Gl_V(N_f)$ corresponds to baryon number conservation and is usually not considered when flavor symmetries are discussed.

What is much more important, though, is the spontaneous breaking of the axial flavor symmetry. From lattice QCD simulations and phenomenological arguments we know that the expectation value

$$\langle \bar\psi\psi \rangle = \langle \bar\psi_R\psi_R \rangle + \langle \bar\psi_L\psi_L \rangle \approx -(240 MeV)^3 \tag{303}$$

in the vacuum state of QCD instead of the symmetric possibility $\langle \bar\psi\psi \rangle = 0$. Phenomenologically, this is known because of the presence of Goldstone modes. The pions are much lighter than the σ mesons. The spontaneous breaking of the axial symmetry also follows from the absence of parity doublets. For example, the pion mass and the δ (or a_0) mass are very different ($m_\pi = 135 MeV$ and $m_\delta = 980 MeV$).

For fermionic quarks there is no need to extend the symmetry group to $Gl_R(N_f) \times Gl_L(N_f)$. In that case we will only consider the usual $SU_R(N_f) \times SU_L(N_f)$ flavor symmetry and it spontaneous breaking to $SU_V(N_f)$. In the case of bosonic quarks we will see in section 3.2 that it is essential to consider the complex extension of $SU(N_f)$. Notice that by extending the symmetry group to complex parameters the number of conserved currents remains the same.

On easily verifies that $\langle \bar\psi\psi \rangle$ is only invariant for $U = V$. The vacuum state thus breaks the chiral symmetry down to $Gl_V(N_f)$. Notice that only the axial symmetries are broken. This is in agreement with the Vafa-Witten theorem [73] which states that vector symmetries cannot be broken spontaneously in vector-like theories such as QCD. We also observe that the *complete* axial group is broken. The reasons behind this maximum breaking [74] of chiral symmetry are less well understood.

9.4. Anti-Unitary Symmetries and the Three-fold Way

The QCD partition function with three or more colors in the fundamental representations has no anti-unitary symmetries. For two colors and fundamental fermions and for adjoint fermions, the Dirac operator has an anti-unitary symmetry which will be discussed below. The classification of QCD according to anti-unitary symmetries was introduced in [18].

9.4.1. QCD with Two Colors

For QCD with two colors in the fundamental representation the gauge field is given by

$$A_\mu = \sum_k a_k \frac{\tau_k}{2}, \tag{304}$$

where the τ_k are the Pauli matrices acting in color space. Because of the pseudo-reality of $SU(2)$ we have that

$$A_\mu^* = -\tau_2 A_\mu \tau_2. \tag{305}$$

Using the explicit representation for the γ-matrices one can easily show that

$$\gamma_\mu^* = \gamma_2 \gamma_4 \gamma_\mu \gamma_2 \gamma_4. \tag{306}$$

For the Dirac operator $iD = i\gamma_\mu \partial_\mu + \gamma_\mu A_\mu$ we thus have

$$\tau_2 \gamma_2 \gamma_4 (iD)^* \tau_2 \gamma_2 \gamma_4 = -iD. \tag{307}$$

This can also be written as

$$[KC\gamma_5 \tau_2, D] = 0, \tag{308}$$

where K is the complex conjugation operator and $C = \gamma_2 \gamma_4$ is the charge conjugation matrix. Because

$$(KC\gamma_5 \tau_2)^2 = 1 \tag{309}$$

we can always find a basis such that the Dirac matrix is real for any A_μ. The proof goes along the same lines as the proof that time reversal symmetry results in real matrix elements for the Hamiltonian in quantum mechanics (see section 2.4).

Exercise. Construct a basis for which D becomes real (without relying on the properties of D other than the anti-unitary symmetry (308) and (309)).

9.4.2. QCD in the Adjoint Representation

For QCD with gauge fields in the adjoint representation the Dirac operator is given by

$$D = \gamma_\mu \partial_\mu \delta_{ab} + f^{abc} \gamma_\mu A_{a\mu}, \tag{310}$$

where the f^{abc} denote the structure constants of the gauge group. Notice that f^{abc} is antisymmetric in its indices so that D is anti-Hermitian. Because of the complex conjugation property of the γ-matrices we have that

$$[\gamma_2 \gamma_4 \gamma_5 K, D] = 0. \tag{311}$$

One easily verifies that in this case

$$(\gamma_2 \gamma_4 \gamma_5 K)^2 = -1 \tag{312}$$

so that the eigenvalues of D are doubly degenerate (see section 2.4). This corresponds to the case $\beta_D = 4$, so that it is possible to construct a basis in which the matrix elements of the Dirac operator can be organized into real quaternions.

10. LOW ENERGY LIMIT OF QCD

In this section we derive the chiral Lagrangian that provides an exact description of QCD at low energies. In the domain where the kinetic term of this chiral Lagrangian can be ignored this Lagrangian constrains the Dirac spectrum to satisfy sum rules for its inverse eigenvalues.

10.1. The chiral Lagrangian

For light quarks the low energy limit of QCD is well understood. It is given by a chiral Lagrangian describing the interactions of the pseudo-scalar mesons. The reason is that pions are Goldstone bosons which are the only light degrees of freedom in a confining theory such as QCD. To lowest order in the quark masses and the momenta, the chiral Lagrangian is completely determined by chiral symmetry and Lorentz invariance. In the case of N_f light quarks with chiral symmetry breaking according to $SU_L(N_f) \times SU_R(N_f) \to SU_V(N_f)$ the Goldstone fields are given by $U \in SU(N_f)$. Under an $SU_L(N_f) \times SU_R(N_f)$ transformation of the quark fields,

$$\begin{aligned} \psi_R &\to U_R \psi_R, \\ \psi_L &\to U_L \psi_L, \end{aligned} \tag{313}$$

the Goldstone fields U transform in the same way as the chiral condensate

$$U \to U_L U U_R^{-1}. \tag{314}$$

The symmetry (313) is broken the mass term. However, the full symmetry can be restored if we also transform the mass term in (300) as

$$M_{RL} \to U_R M_{RL} U_L^{-1}, \qquad M_{LR} \to U_L M_{LR} U_R^{-1}. \tag{315}$$

The low energy effective theory should have the same invariance properties. To second order in the momenta and first order in the quark mass matrix we can write down the following invariant terms:

$$\mathrm{Tr}(\partial_\mu U\, \partial_\mu U^\dagger), \qquad \mathrm{Tr}(M_{RL} U), \qquad \mathrm{Tr}(M_{LR} U^{-1}). \tag{316}$$

Since the QCD partition function is invariant under $M_{RL} \leftrightarrow M_{LR}$, the effective partition function should also have this symmetry. The action of the Goldstone fields is therefore given by the so called Weinberg Lagrangian [75, 76]

$$\mathscr{L}_{\mathrm{eff}}(U) = \frac{F^2}{4}\,\mathrm{Tr}(\partial_\mu U \partial_\mu U^\dagger) - \frac{\Sigma}{2}\,\mathrm{Tr}(M_{RL} U + M_{LR} U^{-1}), \tag{317}$$

where F is the pion decay constant, and Σ is the chiral condensate. The second constant can be identified as the chiral condensate because it is the mean field value of the mass derivative of the partition function. The Goldstone fields can be parametrized as

$$U = \exp(i\sqrt{2}\Pi_a t^a / F), \tag{318}$$

with the generators of $SU(N_f)$ normalized according to $\mathrm{Tr}\, t^a t^b = \delta^{ab}$. This chiral Lagrangian has been used extensively for the description of pion-pion scattering amplitudes [76].

To lowest order in the pion fields we find for equal quark masses

$$\mathscr{L}_{\mathrm{eff}}(U) = \frac{1}{2}\partial_\mu \Pi^a \partial^\mu \Pi^a + \frac{\Sigma m}{F^2}\Pi^a \Pi^a. \tag{319}$$

This results in the pion propagator $1/(p^2 + m_\pi^2)$ with pion mass given by the Gellmann-Oakes-Renner relation

$$m_\pi^2 = \frac{2m\Sigma}{F^2}. \tag{320}$$

In the long-wavelength limit the order of magnitude of the different terms contributing to the action of the chiral Lagrangian is given by [76]

$$S = \int d^4 x \mathscr{L}(U) \sim L^{d-2}\Pi_{NZM}^a{}^2 + L^d \frac{\Sigma m}{F^2}(\Pi_{ZM}^a{}^2 + \Pi_{NZM}^a{}^2). \tag{321}$$

Here, Π_{ZM}^a represents the zero momentum modes with no space time dependence, whereas the nonzero momentum modes are denoted by by $\Pi_{NZM}^a(x)$. This decomposition has two immediate consequences. First, for $\Sigma m/F^2 \gg 1/V$ the fluctuations of the pion fields are small and it is justified to expand U in powers of Π^a. Second, for

$$\frac{\Sigma m}{F^2} \ll \frac{1}{\sqrt{V}} \tag{322}$$

the fluctuations of the zero modes dominate the fluctuations of the nonzero modes, and only the contribution from the zero modes has to be taken into account for the calculation of an observable. In this limit the so called finite volume partition function is given by [77]

$$Z_{N_f}^{\text{eff}}(M, \theta) \sim \int_{U \in SU(N_f)} dU \, e^{V \Sigma \operatorname{Re} \operatorname{Tr} M U e^{i\theta/N_f}}, \tag{323}$$

where the θ-dependence follows from the dependence of the QCD partition function on the combination $m e^{i\theta/N_f}$ only (see section 9.2). We emphasize that any theory with the same pattern of chiral symmetry breaking as QCD can be reduced to the same extreme infrared limit.

The effective partition function at fixed ν follows by Fourier inversion

$$Z_\nu(M) = \frac{1}{2\pi} \int_0^{2\pi} d\theta \, e^{-i\nu\theta} Z(M, \theta). \tag{324}$$

Combining the integral over $SU(N_f)$ and the integral over $U(1)$ and using that

$$\det U e^{i\theta/N_f} = e^{i\theta}, \tag{325}$$

we find that

$$Z_\nu^{\text{eff}}(M) = \int_{U(N_f)} \det^\nu(U) e^{V \Sigma \operatorname{Re} \operatorname{Tr} MU}. \tag{326}$$

We also could have derived this partition function directly from an invariance argument. For field configurations with topological charge ν the difference of the number of right-handed modes and left-handed modes is also equal to ν. Under the transformation (313) and the transformation of the mass matrix (315), the QCD partition in the sector of topological charge ν thus transforms as

$$Z_{\text{QCD}}^\nu \to \det^\nu(U_L U_R^{-1}) Z_{\text{QCD}}^\nu, \tag{327}$$

where U_R and U_L are both $U(N_f)$ transformations. To zeroth order in the momentum and first order in the quark mass, this covariance property immediately gives the partition function (326).

10.2. Leutwyler-Smilga Sum Rules

The Leutwyler-Smilga sum-rules [77] are obtained by expanding the partition function at fixed ν in powers of m before and after averaging over the gauge field configurations and equating the coefficients. This corresponds to an expansion in powers of m of both the QCD partition function and the finite volume partition function (326) in the sector of topological charge ν. As an example, we consider the coefficients of m^2 in the

sector with $v = 0$. We expand the effective partition function to second order in the mass and use the group integrals

$$\int_{U \in U(n)} DU U_{ij}^p = 0,$$
$$\int_{U \in U(n)} DU U_{ij} U_{kl}^{-1} = \frac{1}{n} \delta_{il} \delta_{jk}. \tag{328}$$

One easily verifies that the second integral is consistent with the unitarity relations for U. We then find

$$Z_0^{\text{eff}}(m) = 1 + \frac{\Sigma^2 V^2}{4} m^2. \tag{329}$$

On the other hand, the expansion of the QCD partition function in powers of m is given by

$$\frac{Z_{QCD}^{v=0}(m)}{Z_{QCD}^{v=0}(0)} = 1 + m^2 N_f \langle \sum_{\lambda_k > 0} \frac{1}{\lambda_k^2} \rangle. \tag{330}$$

Equating the coefficients of m^2 results in the Leutwyler-Smilga sum-rule [77]

$$\langle \sum{}' \frac{1}{\lambda_k^2} \rangle = \frac{\Sigma^2 V^2}{4 N_f}, \tag{331}$$

where the prime indicates that the sum is restricted to nonzero positive eigenvalues. *Exercise.* Show that in the sector of topological charge v the sum rule for N_f massless flavors is given by

$$\langle \sum{}' \frac{1}{\lambda_k^2} \rangle = \frac{\Sigma^2 V^2}{4(N_f + v)} \tag{332}$$

By equating higher powers of m^2 one can generate an infinite family of sum-rules for the eigenvalues of the Dirac operator. However, they are not sufficient to determine the Dirac spectrum. The reason is that the mass in the propagator also occurs in the fermion-determinant of the QCD partition function. However, as will be shown in the next chapter, the Dirac spectrum can be obtained from a chiral Lagrangian corresponding to QCD with additional bosonic and fermionic quarks [78]. We conclude that chiral symmetry breaking leads to correlations of the inverse eigenvalues which are determined by the underlying global symmetries.

11. THE DIRAC SPECTRUM IN QCD

In this section we give an interpretation of the Leutwyler-Smilga sum-rules in terms of the smallest eigenvalues of the QCD Dirac operator. We show that the smallest

eigenvalues of the QCD Dirac operator are related to the chiral condensate by means of the Banks-Casher relation.

The order parameter of the chiral phase transition, $\langle \bar{\psi}\psi \rangle$, is nonzero only below the critical temperature or a critical chemical potential. As was shown by Banks and Casher [82], $\langle \bar{\psi}\psi \rangle$ is directly related to the eigenvalue density of the QCD Dirac operator per unit four-volume

$$\Sigma \equiv |\langle \bar{\psi}\psi \rangle| = \lim \frac{\pi \langle \rho(0) \rangle}{V}, \tag{333}$$

where the spectral density of the Dirac operator with eigenvalues $\{\lambda_k\}$ is defined by

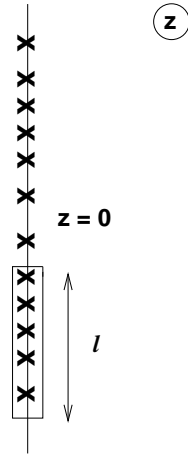$$\rho(\lambda) = \langle \sum_k \delta(\lambda - \lambda_k) \rangle. \tag{334}$$

FIGURE 1. A typical Dirac spectrum. To derive the Banks-Casher relation we integrate the resolvent over the rectangular contour in this figure. (Figure taken from [84].)

Rather than the spectral density we often study the resolvent defined by

$$G(z) = \sum_k \frac{1}{z + i\lambda_k}, \tag{335}$$

which is related to the chiral condensate by

$$\Sigma = - \lim_{m \to 0} \lim_{V \to \infty} G(m)/V. \tag{336}$$

The resolvent can be interpreted as the electric field at z of charges at $i\lambda_k$. Using this analogy it is clear that the resolvent changes sign if z crosses the imaginary axis. Let us look at this in more detail. A typical Dirac spectrum is shown in Fig. 1. The average number of eigenvalues in the rectangular contour in this figure is $\rho(\lambda)l$. Therefore, if we integrate the resolvent along this contour we obtain

$$\oint G(z) = il(G(i\lambda + \varepsilon) - G(i\lambda - \varepsilon)) = 2\pi i \rho(\lambda)l, \tag{337}$$

where the second identity follows from Cauchy's theorem. Using the symmetry of the spectrum we find

$$\text{Re}G(i\lambda + \varepsilon) = \pi\rho(\lambda). \tag{338}$$

Near the center of the spectrum the imaginary part of the resolvent in negligible which immediately results in the Banks-Casher relation (333).

The order of the limits in (333) is important. First we take the thermodynamic limit, next the chiral limit and, finally, the field theory limit.

An important consequence of the Bank-Casher formula (333) is that the eigenvalues near zero virtuality are spaced as

$$\Delta\lambda = 1/\rho(0) = \pi/\Sigma V. \tag{339}$$

For the average position of the smallest nonzero eigenvalue we obtain the estimate

$$\lambda_{\min} = \pi/\Sigma V. \tag{340}$$

This should be contrasted with the eigenvalue spectrum of the non-interacting Dirac operator. Then the eigenvalues are those of a free Dirac particle in a box with eigenvalue spacing equal to $\Delta\lambda \sim 1/V^{1/4}$ for the eigenvalues near $\lambda = 0$. Clearly, the presence of gauge fields leads to a strong modification of the spectrum near zero virtuality. Strong interactions result in the coupling of many degrees of freedom leading to extended states and correlated eigenvalues. Because of asymptotic freedom, the spectral density of the Dirac operator for large λ behaves as $V\lambda^3$. In Fig. 2 we show a plot of a typical average spectral density of the QCD Dirac operator for $\lambda \geq 0$. The spectral density for negative λ is obtained by reflection with respect to the y-axis. More discussion of this figure will be given in section 12.5.
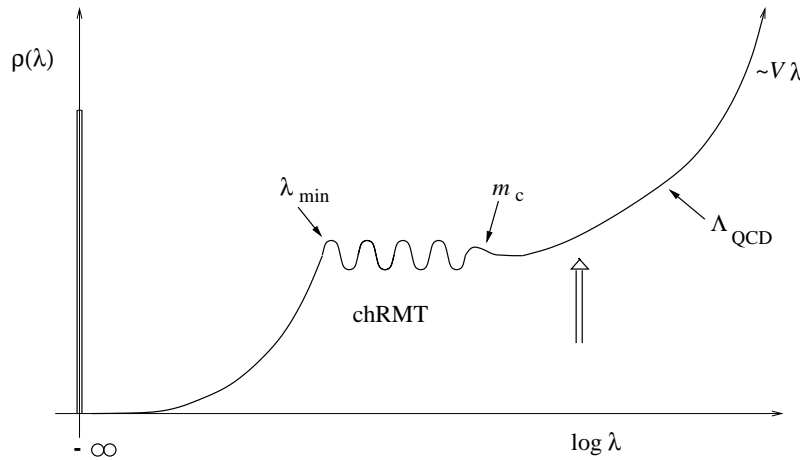

FIGURE 2. Schematic picture of the average spectral density of QCD Dirac operator. (Taken from ref. [83].)

Let us study the Leutwyler-Smilga sum rule for equally spaced eigenvalues, i.e.

$$\lambda_n = \frac{\pi n}{\Sigma V}. \tag{341}$$

This results in the sum

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} = \frac{\Sigma^2 V^2}{6}.$$

(342)

We obtain the right parametric dependence of the Leutwyler-Smilga sum rules. The density of the smallest eigenvalues is suppressed by the number of massless quarks through the fermion determinant. Therefore it is not surprising that the exact result decreases with N_f.

Because the eigenvalues near zero are spaced as $\sim 1/\Sigma V$ it is natural to introduce the microscopic variable

$$u = \lambda V \Sigma,$$

(343)

and the microscopic spectral density [17]

$$\rho_s(u) = \lim_{V \to \infty} \frac{1}{V\Sigma} \rho\left(\frac{u}{V\Sigma}\right).$$

(344)

We expect that this limit exists and converges to a universal function which is determined by the global symmetries of the QCD Dirac operator. The calculation of this universal function from QCD is the main objective of the next chapter. We will calculate $\rho_s(u)$ both from the simplest theory in this universality class which is chiral Random Matrix Theory (chRMT) and from the partial quenched chiral Lagrangian which describes the low energy limit of the QCD partition function. We find that the two results coincide below the Thouless energy. Universality can also directly be analyzed within the framework of of Random Matrix Theory. This can be done by showing that the microscopic spectral density is insensitive to deformations of the probability distribution. The interested reader is referred to the extensive literature on this subject [85, 86, 87, 88, 89, 90, 91, 92].

12. PARTIAL QUENCHING AND THE RESOLVENT OF THE QCD DIRAC OPERATOR

In this chapter we study the spectrum of the QCD Dirac operator by means of the resolvent. For simplicity we will only consider the case with three or more colors and fundamental fermions.

12.1. The resolvent in QCD

In terms of the eigenvalues of the Dirac operator the resolvent is defined as [81]

$$\begin{aligned}
G(z; m_1, \cdots, m_{N_f}) &= \frac{1}{V} \sum_k \left\langle \frac{1}{i\lambda_k + z} \right\rangle \\
&= \frac{1}{V} \int d\lambda \frac{\rho(\lambda; m_1, \cdots, m_{N_f})}{i\lambda + z}.
\end{aligned}$$

(345)

Here, $\langle \cdots \rangle$ denotes an average with respect to the distribution of the eigenvalues. Sometimes the argument z of the resolvent is referred to as the valence quark mass [93]. It does not enter in the fermion determinant as a regular quark and is therefore a direct probe of the Dirac spectrum.

The relation (345) can then be inverted to give $\rho(\lambda; m_1, \ldots, m_{N_f})$. As mentioned in [78], the spectral density follows from the discontinuity across the imaginary axis,

$$\text{Disc}|_{z=i\lambda} \, G(z) = \lim_{\varepsilon \to 0} G(i\lambda + \varepsilon) - G(i\lambda - \varepsilon) = \frac{2\pi}{V} \sum_k \langle \delta(\lambda + \lambda_k) \rangle = \frac{2\pi}{V} \rho(\lambda),$$
(346)

where we have suppressed the dependence on the sea-quark masses. This relation follows by writing $G(z)$ as a sum over eigenvalues.

12.2. Generating function

In this section we introduce a generating function for the resolvent. Similarly to the supersymmetric method discussed before this achieved by introducing the ratio of two determinants in the partition function. This corresponds to a Euclidean partition function of the form

$$Z_\nu^{\text{pq}}(z, z', m_f) \; = \; \int dA \, \frac{\det(D+z)}{\det(D+z')} \prod_{f=1}^{N_f} \det(D+m_f) \, e^{-S_{YM}} \, ,$$
(347)

which we will call the partially quenched or pq-QCD partition function. For $z = z'$ this partition function coincides with the original QCD partition function. However, it is also the generator of the resolvent (see (345)). In the sector of topological charge ν we find

$$G(z; m_1, \cdots, m_{N_f}) = \frac{1}{V} \left. \frac{\partial}{\partial z} \right|_{z=z'} \log Z_\nu^{\text{pq}}(z, z', m_f).$$
(348)

In addition to the regular quarks, the partition function (347) has additional bosonic and fermionic ghost quarks. Our aim is to find the chiral Lagrangian corresponding to (347). If we are successful, we have succeeded in deriving a partition function for the extreme infrared limit of the spectrum of the QCD Dirac operator. The derivation of this partition function and its domain of validity will be discussed in the next sections. We will first discuss the subtleties that arise for the axial symmetry with bosonic quarks.

12.3. Bosonic Quarks

For bosonic quarks the Goldstone bosons cannot be parameterized by a unitary matrix. The reason is that symmetry transformations have to be consistent with the convergence of the bosonic integrals. Let us consider the case of one bosonic flavor. Then

$$\det^{-1} \begin{pmatrix} m & id \\ id^\dagger & m \end{pmatrix} = \frac{1}{\pi^2} \int d^2\phi_1 d^2\phi_2 \exp \left[- \begin{pmatrix} \phi_1^* \\ \phi_2^* \end{pmatrix} \begin{pmatrix} m & id \\ id^\dagger & m \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right], \quad (349)$$

so that the exponent is purely imaginary for $m = 0$ and convergent for $\text{Re}(m) > 0$.

The most general flavor symmetry transformation of the action in (349) is $Gl(2)$ which can be parameterized as

$$U = e^H V \quad \text{with} \quad H^\dagger = H \quad \text{and} \quad VV^\dagger = 1. \tag{350}$$

For U to be a symmetry transformation for $m = 0$ we require that

$$U^\dagger \begin{pmatrix} 0 & id \\ id^\dagger & 0 \end{pmatrix} U = \begin{pmatrix} 0 & id \\ id^\dagger & 0 \end{pmatrix}, \tag{351}$$

so that H has to be a multiple of σ_3 and V has to be a multiple of the identity. The V transformations in (350) are not broken by the mass term and therefore represent the vector symmetry. Only the symmetry transformation $\exp(s\sigma_3)$ is broken by the mass term so that the axial transformations are parameterized by

$$U = \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix} \qquad \text{with} \quad s \in \langle -\infty, \infty \rangle. \tag{352}$$

For N_f bosonic flavors the axial transformations are parameterized by

$$U = \begin{pmatrix} e^H & 0 \\ 0 & e^{-H} \end{pmatrix} \qquad \text{with} \qquad H^\dagger = H, \tag{353}$$

which is the coset $Gl(N_f)/U(N_f)$.

Also in the case of bosonic quarks the axial symmetry is spontaneously broken:

$$\langle \phi_1^* \phi_1 \rangle + \langle \phi_2^* \phi_2 \rangle = \langle \bar\psi_R \psi_L \rangle + \langle \bar\psi_L \psi_R \rangle. \tag{354}$$

The reason is that both expectation values are equal to the inverse trace of the Dirac operator.

12.4. The infrared limit of QCD

Ignoring for the moment convergence questions the global flavor symmetry of (347) is given by

$$Gl_R(N_f + 1|1) \times Gl_L(N_f + 1|1). \tag{355}$$

We already have seen that the axial symmetry for bosonic quarks is not $U(N_f)$ but rather $Gl(N_f)/U(N_f)$. Although the axial flavor symmetry group of the fermionic quarks is not a priori determined by convergence requirements we will see in this section that supersymmetry necessarily imposes that this symmetry group is compact, i.e. equal to $U(N_f)$.

Under a $Gl_R(N_f + 1|1) \times Gl_L(N_f + 1|1)$ transformation the quarks fields transforms as

$$\begin{aligned} \psi_R &\to U_R \psi_R, & \psi_L &\to U_L \psi_L \\ \bar\psi_R &\to \bar\psi_R U_R^{-1}, & \bar\psi_L &\to \psi_L U_L^{-1}. \end{aligned} \tag{356}$$

The subscripts refer to the right-handed (R) or left-handed (L) chirality of the quarks. The latter transformations transform the right-handed and the left-handed fermion fields in the same way.

For $M = 0$ and $\nu = 0$ this is a symmetry of the QCD action. At nonzero mass this symmetry can be restored if we also transform the mass term according to

$$M_{RL} \rightarrow U_R M_{RL} U_L^{-1},$$
$$M_{LR} \rightarrow U_L M_{LR} U_R^{-1}. \tag{357}$$

In the sector of topological charge ν the partially quenched partition function transforms as

$$Z_\nu(M_{RL}, M_{LR}) \rightarrow \text{Sdet}^\nu U_R U_L^{-1} Z_\nu(U_R M_{RL} U_L^{-1}, U_L M_{LR} U_R^{-1}) \tag{358}$$

The low energy effective partition function should have the same covariance properties as the full QCD partition function.

Because the expectation value of both the condensate of the bosonic quarks and the condensate of the fermionic quarks is nonzero in the chirally broken phase, the flavor symmetry of the partially quenched QCD partition function (347) is broken spontaneously according to

$$Gl_R(N_f + 1|1) \times Gl_L(N_f + 1|1) \rightarrow Gl_V(N_f + 1|1). \tag{359}$$

Therefore the low energy limit of the partially quenched QCD partition function is a theory of weakly interacting Goldstone bosons parameterized by

$$Gl_R(N_f + 1|1) \times Gl_L(N_f + 1|1) / Gl_V(N_f + 1|1) \equiv Gl_A(N_f + 1|1). \tag{360}$$

For a confining theory such as QCD the only low lying modes are the Goldstone modes associated with the spontaneous breaking of chiral symmetry. However, the Goldstone manifold is not the full group $Gl_A(N_f + 1|1)$. As we already discussed for the case of bosonic quarks the $U_A(1)$ transformation in which the bosonic fields with index 1 and 2 are transformed according to an opposite phase factor is not a symmetry of the partition function. What is a symmetry of the partition function is the axial transformation

$$\begin{aligned} \phi_1 &\rightarrow e^s \phi_1, & \phi_1^* &\rightarrow e^s \phi_1^*, \\ \phi_2^* &\rightarrow e^{-s} \phi_2^* & \phi_2 &\rightarrow e^{-s} \phi_2. \end{aligned} \tag{361}$$

Mathematically, this symmetry group is $Gl(1)/U(1)$. Had we restricted ourselves to the unitary subgroup $U(N_f + 1|1)$ of $Gl(N_f + 1|1)$ from the start, we would have missed this class of symmetry transformations.

Let us now consider the low energy limit of the partially quenched QCD partition function in the sector of topological charge ν. Under a $Gl_R(N_f + 1|1) \times Gl_L(N_f + 1|1)$ transformation the Goldstone modes transform as

$$Q \rightarrow U_R Q U_L^{-1}. \tag{362}$$

The low energy effective partition function should have the same transformation properties a the partially quenched partition function To lowest order in the mass and the momentum we can write down the following invariants

$$\mathrm{Str}\partial_\mu Q^{-1}\partial_\mu Q \quad \text{and} \quad \mathrm{Str}(M_{RL}Q) + \mathrm{Str}(M_{LR}Q^{-1}). \tag{363}$$

If we factorize the Goldstone fields into the zero momentum modes Q_0 and the nonzero momentum modes $Q(x)$ as

$$Q = Q_0 Q(x), \tag{364}$$

the low energy effective partition function that transforms in the same way as the partially quenched QCD partition function (358) is given by

$$Z_\nu(M) = \int_{Q \in Gl(N_f+1|1)} dQ \mathrm{Sdet}^\nu(Q_0) e^{-\int d^4 x \mathscr{L}^{\mathrm{eff}}(Q)}, \tag{365}$$

where

$$\mathscr{L}^{\mathrm{eff}}(Q) = \frac{F^2}{4}\mathrm{Str}\partial_\mu Q^{-1}\partial_\mu Q + \frac{1}{2}\Sigma\mathrm{Str}(M_{RL}Q) + \frac{1}{2}\Sigma\mathrm{Str}(M_{LR}Q^{-1}). \tag{366}$$

We already have seen that the boson-boson block of $Gl(N_f+1|1)$ is $Gl(1)/U(1)$. If we parameterize the field Q as

$$Q = e^{\sum_k T_k \pi_k / F}, \tag{367}$$

with T_k the generators of $G(N_f+1|1)$, to second order in the Goldstone fields the mass term is given by

$$\frac{\Sigma}{2F^2}\mathrm{Str}(M \sum_k T_k^2 \pi_k^2). \tag{368}$$

Let us take all (ghost-)quark masses positive. Because of the supertrace there is a relative minus sign between boson-boson Goldstone modes and fermion-fermion Goldstone modes. The boson-boson Goldstone modes are noncompact and require that the overall minus sign of the mass term is negative. In order to avoid tachyonic fermion-fermion Goldstone modes, we have to compensate the minus sign of the supertrace. This can be done by choosing the parameters multiplying the fermion-fermion generators purely imaginary. This corresponds to a compact parametrization of the fermion-fermion Goldstone manifold. This integration manifold is the maximum Riemannian submanifold [44, 78, 79] of $Gl(N_f+1|1)$ and will be denoted as $\hat{Gl}(N_f+1|1)$.

12.5. Domains in (Partially-Quenched) Chiral Perturbation Theory

In chiral perturbation theory the different domains of validity where analyzed in detail by Gasser and Leutwyler [80]. A similar analysis applies to partially quenched chiral perturbation theory [81]. The idea is as follows. The Q field can be decomposed as [80]

$$Q = Q_0 e^{i\psi(x)}. \tag{369}$$

where Q_0 is a constant (zero-momentum) field. The kinetic term for the ψ fields is given by

$$\frac{1}{2}\partial_\mu \psi^a(x)\partial_\mu \psi^a(x) \sim L^{-2}\psi^2(x).$$ (370)

We observe that the magnitude of the fluctuations of the ψ field are of order $1/L$ which justifies a perturbative expansion of $\exp(i\psi(x))$. The fluctuations of the zero modes, on the other hand, are only limited by the mass term

$$\frac{1}{2}V\Sigma\text{Str}M(Q_0+Q_0^{-1}).$$ (371)

For quark masses $m \gg 1/V\Sigma$, the field U_0 fluctuates close to the identity and the U_0 field can be expanded around the identity as well. If $m \ll \Lambda_{\text{QCD}}$ we are in the domain of chiral perturbation theory.

The same arguments apply to the partially quenched chiral Lagrangian. However, there is an important difference. The mass of the ghost-quarks is an external parameter that can take on any value we wish. The mass of the Goldstone modes containing these quarks is given by

$$M_{zz} = \frac{2z\Sigma}{F^2}.$$ (372)

Therefore, independent of the quark masses there is always a domain where the fluctuations of the zero momentum modes dominate the fluctuations of the nonzero momentum modes. This domain is given by [81]

$$z \ll \frac{F^2}{\Sigma L^2}$$ (373)

and is known as the ergodic domain of QCD. Sometimes it is also referred to as the epsilon regime[1]. In order that the non-Goldstone modes do not contribute to the partition function we have to require that

$$L \gg \frac{1}{\Lambda_{\text{QCD}}}.$$ (374)

In the Dirac spectrum we can distinguish three important scales. First, the scale of the smallest eigenvalue,

$$\lambda_{\text{min}} = \frac{\pi}{\Sigma V}.$$ (375)

---

[1] The epsilon regime is the regime [80] where $m_\pi \sim O(\varepsilon^2)$ and $1/L \sim O(\varepsilon)$ in an expansion in powers of ε of the chiral Lagrangian. These conditions are more strict than the inequality (373). For example, we are still in the ergodic domain if $1/L \sim O(\varepsilon^{3/2})$

Second, the valence quark mass corresponding to a valence quark Goldstone boson with Compton wavelength equal to the size of the box

$$m_c = \frac{F^2}{\Sigma L^2}.$$ (376)

Third, the scale of Λ_{QCD} which sets the mass scale of QCD. Based on these scales we can distinguish four different domains. In the domain where z is of the order of $\lambda_{\min}$ or less we can restrict ourselves to the zero momentum sector of the theory, but we have to take into account quantum fluctuations to all orders. Also for $\lambda_{\min} \ll z \ll m_c$, we only have to include the zero momentum modes but in this case the quantum fluctuations can be treated semi-classically, or in field theory language, by a loop expansion. The time scale conjugate to m_c is of the order of the diffusion time across the length of the box which explains the name the ergodic domain. For $m_c \ll z \ll \Lambda_{QCD}$, chiral perturbation theory still applies, but the zero momentum modes no longer dominate the partition function. For $z \gg \Lambda_{QCD}$, the masses of the Goldstone modes and the other hadronic excitations are of the same order of magnitude. Chiral perturbation theory is no longer applicable to the spectrum of the Dirac operator, and one has to take into account the full QCD partition function. A schematic drawing of the Dirac spectrum in the different domains is given in Fig. 4.

In the theory of disordered mesoscopic systems the scale below which random matrix theory is valid is known as the Thouless energy and is given by [94, 95]

$$E_c = \frac{\hbar D}{L^2},$$ (377)

where D is the diffusion constant for the diffusive motion of electrons in a disordered sample. The time conjugate to E_c is the time scale over which an electron diffuses across the sample. The time scale in mesoscopic physics corresponding to Λ_{QCD} is the elastic scattering time. The domain in between E_c and $\hbar/\tau_e$, where τ_e is the elastic scattering time, is known as the diffusive domain. This domain is characterized by diffusive motion of electrons in the disordered sample.

In the derivation of the zero momentum limit of the partially quenched QCD partition function we have only relied on symmetries and the spontaneous breaking of symmetries. In particular, this means that any theory with the same symmetry breaking pattern and only Goldstone modes as light particles will lead to the same low energy effective theory. The simplest theory in this class is chiral Random Matrix Theory introduced in chapter 2.2. This theory has the flavor symmetries of QCD and, in the limit of large matrices, the non-Goldstone modes decouple from the theory. Therefore, the zero momentum properties of QCD can be derived from chiral Random Matrix Theory.

12.6. Zero momentum limit of the effective partition function

Since z is a free parameter, independent of the quark masses, we can always find a part of the Dirac spectrum inside the domain (373). In this domain the zero momentum

limit of the QCD partition function in the sector of topological charge v is given by [78]

$$Z_{\text{eff}}^v(M) = \int_{Q \in \hat{Gl}(N_f+1|1)} dQ \, \text{Sdet}^v Q \, e^{V \frac{\Sigma}{2} \text{Str}(MQ+MQ^{-1})}. \tag{378}$$

The number of QCD Dirac eigenvalues that is described by this partition function is of the order (see (339))

$$\frac{F^2}{\Sigma L^2 \Delta \lambda} = F^2 L^2. \tag{379}$$

This number increases linearly in N_c for $N_c \to \infty$ which was studied recently by lattice simulations [96].

In the next section we will study this partition function in the quenched limit ($N_f = 0$) and show that the resolvent coincides with the result obtained from chiral Random Matrix Theory [78, 79]. The Random Matrix result will be derived by means of the supersymmetric method in section 12.8.

12.7. Nonperturbative evaluation of $G(z)$ in the quenched limit

In this section we evaluate the resolvent of QCD for the simplest case of $N_f = 0$ and $v = 0$ in the domain $z \ll F^2/\Sigma L^2$. In this domain the partition function is given by

$$Z(J) = \int_{Q \in \hat{Gl}(1|1)} dU \exp\left[\frac{\Sigma V}{2} \text{Str}\begin{pmatrix} z+J & 0 \\ 0 & z \end{pmatrix}(Q+Q^{-1})\right], \tag{380}$$

where the integration is over the maximum super-Riemannian sub-manifold of $Gl(1|1)$. This manifold is parametrized by [79]

$$Q = \exp\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}\begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^s \end{pmatrix}. \tag{381}$$

The integration measure is the Haar measure which in terms of this parameterization it is given by

$$\text{Sdet}\frac{\delta Q_{kl}}{\delta\phi\,\delta s\,\delta\alpha\,\delta\beta}\,d\alpha d\beta d\phi ds, \tag{382}$$

where $\delta Q \equiv Q^{-1}dQ$.

It is straightforward to calculate the Berezinian going from the variables $\{\delta Q_{11}, \delta Q_{22}, \delta Q_{12}, \delta Q_{21}\}$ to the variables $\{\delta\phi, \delta s, \delta\alpha, \delta\beta\}$. The derivative matrix is given by

$$B = \frac{\delta Q_{kl}}{\delta\phi\,\delta s\,\delta\alpha\,\delta\beta} = \begin{pmatrix} i & 0 & \frac{\beta}{2} & \frac{\alpha}{2} \\ 0 & 1 & \frac{\beta}{2} & \frac{\alpha}{2} \\ 0 & 0 & e^{s-i\phi} & 0 \\ 0 & 0 & 0 & e^{-s+i\phi} \end{pmatrix}. \tag{383}$$

Using the definition of the graded determinant one simply finds that $\text{Sdet}B = i$. Up to a constant, the integration measure is thus given by $d\phi ds d\alpha d\beta$. In general, for $N_f \neq 0$, the Berezinian is more complicated [79].

We also need

$$\frac{1}{2}(Q + Q^{-1}) = \begin{pmatrix} \cos\phi(1 + \frac{\alpha\beta}{2}) & \alpha(e^s - e^{-i\phi}) \\ \beta(e^{i\phi} - e^{-s}) & \cosh s(1 - \frac{\alpha\beta}{2}) \end{pmatrix}. \tag{384}$$

After differentiating with respect to the source term $(G(z) = \partial_J \log Z(J)|_{J=0})$ this results in

$$\frac{G(z)}{V\Sigma} = \int \frac{d\phi ds d\alpha d\beta}{2\pi} \cos\phi(1 + \frac{\alpha\beta}{2}) e^{x\cos\phi(1+\frac{\alpha\beta}{2}) - x\cosh s(1-\frac{\alpha\beta}{2})}. \tag{385}$$

With the Grassmann integral given by the coefficient of $\alpha\beta$ we obtain

$$\frac{G(z)}{V\Sigma} = \int \frac{ds d\phi}{4\pi} [\cos\phi + x(\cos\phi + \cosh s)\cos\phi] e^{x(\cos\phi - \cosh s)}.$$

All integrals can be expressed in terms of modified Bessel functions. We find [78, 79]

$$\frac{G(z)}{V\Sigma} = I_1(x)K_0(x) + \frac{x}{2}(I_2(x)K_0(x) + I_0(x)K_0(x) + 2I_1(x)K_1(x)). \tag{386}$$

After using the recursion relation for modified Bessel functions,

$$I_2(x) = I_0(x) - \frac{2}{x}I_1(x), \tag{387}$$

we arrive at the final result [81, 78, 79]

$$\frac{G(z)}{V\Sigma} = x(I_0(x)K_0(x) + I_1(x)K_1(x)) \tag{388}$$

where $x = zV\Sigma$.

This calculation can be generalized to arbitrary N_f and arbitrary ν. The calculation for arbitrary N_f is much more complicated, but with a natural generalization of the factorized parameterization, and using some known integrals over the unitary group, one arrives at the following expression in terms of modified Bessel functions

$$\frac{G(z)}{\Sigma} = x(I_a(x)K_a(x) + I_{a+1}(x)K_{a-1}(x)), \tag{389}$$

where $a = N_f + |\nu|$. This result is in complete agreement with the microscopic spectral density derived from chRMT [81].

For $a = 0$ this result is plotted in Fig. 3. We observe that, below some scale, lattice QCD data obtained by the Columbia group [97] closely follow this curve.
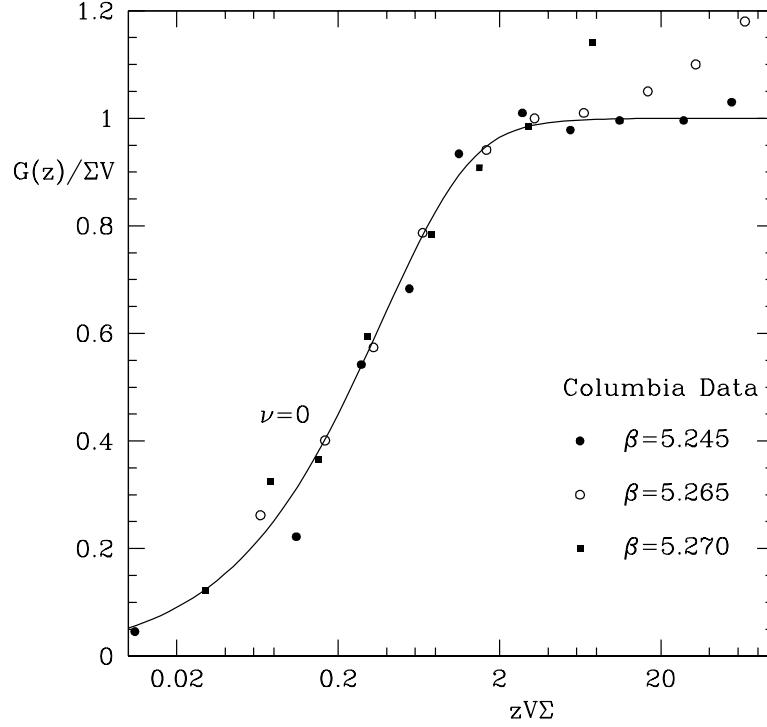
FIGURE 3. The resolvent of quenched QCD. The points represent lattice data obtained by the Columbia group, and the theoretical prediction (388) is given by the solid curve. (Taken from ref. [84].)

12.8. Random Matrix Calculation of $G(z)$ for the chGUE

In this section we calculate the one-point function of the chiral Gaussian Unitary Ensemble by means of the supersymmetric method [98, 99, 85, 88]. The starting point is the partition function

$$Z(z, z+J) = \int \mathscr{D}W \; \frac{\det(z+J+D)}{\det(z+D)} \; \exp[-n\Sigma^2 \mathrm{Tr}\, WW^\dagger],$$

$$(390)$$

where D is given by the anti-hermitian random matrix

$$D = \begin{pmatrix} 0 & iW \\ iW^\dagger & 0 \end{pmatrix} . \tag{391}$$

In this section we only consider the case of zero topological charge where W is an $n \times n$ complex matrix. The integral is over the real and imaginary parts of the matrix elements of W. For definiteness we take the real part of z to be positive. The resolvent is obtained by differentiating the partition function

$$G(z) = \frac{1}{V}\mathrm{Tr}\left\langle \frac{1}{z+D} \right\rangle = \frac{1}{V}\left.\frac{\partial \log Z(z,z+J)}{\partial J}\right|_{J=0} , \tag{392}$$

The determinant in the partition function is written as an integral over Grassmann variables χ and χ^*:

$$\begin{aligned}
\det(z+J+D) &= \int \prod_{i=1}^{n} d\chi_{1i}^* d\chi_{1i} \prod_{i=1}^{n} d\chi_{2i}^* d\chi_{2i} \\
&\times \exp -\begin{pmatrix} \chi_1^* \\ \chi_2^* \end{pmatrix} \begin{pmatrix} z+J & iW \\ iW^\dagger & z+J \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} . \tag{393}
\end{aligned}$$

Similarly, the inverse determinant can be written as an integral over the commuting variables ϕ and ϕ^*:

$$\begin{aligned}
\det^{-1}(z+D) &= \frac{1}{(-2\pi i)^{2n}} \int \prod_{i=1}^{n} d\phi_{1i} d\phi_{1i}^* \prod_{i=1}^{n} d\phi_{2i} d\phi_{2i}^* \\
&\times \exp -\begin{pmatrix} \phi_1^* \\ \phi_2^* \end{pmatrix} \begin{pmatrix} z & iW \\ iW^\dagger & z \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} . \tag{394}
\end{aligned}$$

After performing the Gaussian integral over W we obtain

$$\begin{aligned}
Z(z,z+J) &= \int \prod_{i=1}^{n} d\phi_{1i} d\phi_{1i}^* d\chi_{1i}^* d\chi_{1i} \prod_{i=1}^{n} d\phi_{2i} d\phi_{2i}^* d\chi_{2i}^* d\chi_{2i} \\
&\times \exp\left[-\begin{pmatrix} \phi_1^* \\ \phi_2^* \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} - \begin{pmatrix} \chi_1^* \\ \chi_2^* \end{pmatrix} \begin{pmatrix} z+J & 0 \\ 0 & z+J \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}\right] \\
&\times \exp -\frac{1}{n\Sigma^2}(\chi_{2j}^* \chi_{1i} + \phi_{2j}^* \phi_{1i})(\chi_{1i}^* \chi_{2j} + \phi_{1i}^* \phi_{2j}) . \tag{395}
\end{aligned}$$

The terms of fourth order in the fields are decoupled to second order terms by means of the Hubbard-Stratonovitch transformation. There are no convergence problems for the terms that contain Grassmann variables. In these cases we use the identity

$$e^{\chi_1^* \cdot \chi_1 \chi_2^* \cdot \chi_2/n\Sigma^2} = c_1 \int d\rho_1 d\rho_2 \exp[-n\Sigma^2(\rho_1^2 + \rho_2^2) - (\rho_1 - i\rho_2)\chi_1^* \cdot \chi_1 - (\rho_1 + i\rho_2)\chi_2^* \cdot \chi_2] , \tag{396}$$

for the term of fourth order in the Grassmann variables and

$$
\begin{aligned}
e^{-\chi_1^* \cdot \phi_1 \, \chi_2 \cdot \phi_2^* / n\Sigma^2} &= c_2 \int d\alpha^* d\beta \, \exp[-n\Sigma^2 \alpha^* \beta + \alpha^* \chi_1^* \cdot \phi_1 - \beta \phi_2^* \cdot \chi_2] , \\
e^{\chi_1 \cdot \phi_1^* \, \chi_2^* \cdot \phi_2 / n\Sigma^2} &= c_2 \int d\beta^* d\alpha \, \exp[-n\Sigma^2 \beta^* \alpha - \alpha \chi_1 \cdot \phi_1^* + \beta^* \phi_2 \cdot \chi_2^*] ,
\end{aligned}
$$

$$(397)$$

for the mixed terms. In the above equations, ρ_1 and ρ_2 are commuting variables and α, α^*, β and β^* are Grassmann variables. The constants c_1 and c_2 are not necessary for the calculation of the partition function. We can always fix the overall normalization constant from the requirement that $Z(z, z) = 1$. In the case of the the four boson term we have to be more careful. This term can be rewritten as

$$
e^{-\phi_1^* \cdot \phi_1 \, \phi_2 \cdot \phi_2^* / n\Sigma^2} = \exp -\frac{1}{4n\Sigma^2} [(\phi_1^* \cdot \phi_1 + \phi_2 \cdot \phi_2^*)^2 - (\phi_1^* \cdot \phi_1 - \phi_2 \cdot \phi_2^*)^2]. \qquad (398)
$$

The uniform convergence of the ϕ_i-integrations necessary to interchange the order of the integrations can be achieved if we perform a Hubbard-Stratonovitch transformation using the identity [88]

$$
e^{-a^2 + b^2} = \frac{-n\Sigma^2}{\pi i} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} \sigma d\sigma e^{-n\Sigma^2 \sigma^2 - 2i\sqrt{n}\Sigma a\sigma \cosh s - 2i\sqrt{n}\Sigma b\sigma \sinh s} , \qquad (399)
$$

where a is real positive and $b^2 - a^2$ has a negative real part. Indeed, these conditions are fulfilled if we choose

$$
a = \frac{1}{2\Sigma\sqrt{n}} (\phi_1^* \cdot \phi_1 + \phi_2 \cdot \phi_2^*), \qquad b = \frac{1}{2\Sigma\sqrt{n}} (\phi_1^* \cdot \phi_1 - \phi_2 \cdot \phi_2^*). \qquad (400)
$$

An alternative way to find correct transformation of the four-boson term is to use the Ingham-Siegel integral instead of the Hubbard-Stratonovitch transformation [100, 101].

Combining the above results we obtain

$$
\begin{aligned}
Z(z, z + J) = \; & c \int d\phi_1 d\phi_1^* d\phi_2 d\phi_2^* d\chi_1 d\chi_1^* d\chi_2 d\chi_2^* \, d[\sigma] \exp\left[-n\Sigma^2(\sigma^2 + \rho_1^2 + \rho_2^2 + \alpha^*\beta + \beta^*\alpha)\right] \\
& \times \; \exp - \begin{pmatrix} \phi_1^* \\ \phi_2^* \\ \chi_1^* \\ \chi_2^* \end{pmatrix} \begin{pmatrix} z + i\sigma e^s & 0 & \alpha & 0 \\ 0 & z + i\sigma e^{-s} & 0 & \beta \\ \alpha^* & 0 & z + J + \rho_1 - i\rho_2 & 0 \\ 0 & \beta^* & 0 & z + \rho_1 + \rho_2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \chi_1 \\ \chi_2 \end{pmatrix} ,
\end{aligned}
$$

$$(401)$$

where

$$
d[\sigma] = \sigma d\sigma ds \, d\rho_1 \, d\rho_2 \, d\alpha \, d\alpha^* \, d\beta \, d\beta^* . \qquad (402)
$$

The Gaussian integrals over the ϕ and χ variables are equal to the product of inverse graded determinants. Writing out the 2×2 graded determinants according to (76) we

obtain

$$Z(z, z+J) = \int d[\sigma] \exp\left[-n\Sigma^2(\sigma^2 + \rho_1^2 + \rho_2^2 + \alpha^*\beta + \beta^*\alpha)\right]$$
$$\times (z+J+\rho_1-i\rho_2)^n(z+J+\rho_1+i\rho_2)^n$$
$$\times (z+i\sigma e^s + \frac{\alpha\alpha^*}{z+J+\rho_1-i\rho_2})^{-n}(z+i\sigma e^{-s} + \frac{\beta\beta^*}{z+J+\rho_1+i\rho_2})^{-n}.$$
$$(403)$$

The Grassmann integrals can be performed trivially by expanding the exponent and the pre-exponent. The expression for the partition function can be further simplified by shifting $\sigma \cosh s$ by iz and ρ_1 by $-z-J$. Using the polar coordinates

$$\rho_1 = -z - J + \rho\cos\phi/\Sigma,$$
$$\rho_2 = \rho\sin\phi/\Sigma, \qquad (404)$$

and rescaling σ by $1/\Sigma$ we find

$$Z(z, z+J) = c \int_{-\infty}^{\infty} \sigma d\sigma \int_{-\infty}^{\infty} ds \int_0^{\infty} \rho d\rho \int_0^{2\pi} d\varphi \left[1 + \frac{1}{\sigma^2\rho^2}\right]$$
$$\times \left(\frac{\rho^2}{-\sigma^2}\right)^n e^{-n(\sigma^2 + \rho^2 + 2iz\Sigma\sigma\cosh s - 2(z+J)\Sigma\rho\cos\varphi + \Sigma^2((z+J)^2 - z^2))}. (405)$$

Because of the shift of the σ-variable its integration path lies below the real axis.

The partition function (405) is valid for any n. In the large n limit the ρ and the σ integrations can be performed by a saddle point approximation. The saddle point equations are given by

$$\rho^2 = 1, \qquad \sigma^2 = -1, \qquad (406)$$

leading to the saddle points

$$\bar{\rho} = \pm 1, \qquad \bar{\sigma} = \pm i. \qquad (407)$$

The saddle point $\bar{\rho} = -1$ and and $\bar{\sigma} = -i$ are not accessible (remember that σ contains an infinitesimal increment). By expanding $\rho = 1 + \delta\rho$ and $\sigma = 1 + \delta\sigma$ to second order about the saddle point we obtain

$$Z(z, z+J) = c \int_{-\infty}^{\infty} d\delta\sigma \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} d\delta\rho \int_0^{2\pi} d\varphi$$
$$\times \left[2z\Sigma\cosh s(\delta\sigma)^2 - 2(z+J)\Sigma\cos\varphi(\delta\rho)^2 - i(\delta\rho^2) + i(\delta\sigma)^2\right]$$
$$\times e^{-2n((\delta\sigma)^2 + (\delta\rho)^2) + z\Sigma\cosh s - (z+J)\Sigma\cos\varphi + \Sigma^2((z+J)^2 - z^2))}, \qquad (408)$$

The terms that are odd in either $\delta\rho$ or $\delta\sigma$ integrate to zero and have not been written down. The Gaussian integrals over $\delta\rho$ and $\delta\phi$ are easily calculated. As final result for

the partition function we find

$$Z(z, z+J) = \frac{n}{\pi} \int_{-\infty}^{\infty} ds \int_{0}^{2\pi} d\varphi$$
$$\times \; [(z+J)\Sigma \cosh s - z\Sigma \cos \varphi] \, e^{-2n(z\Sigma \cosh s - z\Sigma \cos \varphi + \Sigma^2((z+J)^2 - z^2))} \,,$$
$$(409)$$

The normalization constant in (408) has been chosen such that $Z(z, z) = 1$. The integrals can be expressed in terms of modified Bessel functions. This results in

$$Z(z, z+J) = 2n\Sigma(z+J)K_0(2n\Sigma z)I_1(2n\Sigma(z+J)) + 2n\Sigma z K_1(2n\Sigma z)I_0(2n\Sigma(z+J)). \quad (410)$$

The normalized resolvent follows after differentiation with respect to the source term. It is given by

$$\frac{G(z)}{\Sigma} = x(K_0(x)I_0(x) + K_1(x)I_1(x)) \qquad \text{with} \qquad x = 2n\Sigma z, \qquad (411)$$

which we have obtained previously from the low energy effective partition function (see eq. (389)).

13. INTEGRABILITY AND THE REPLICA LIMIT OF LOW ENERGY QCD PARTITION FUNCTION

In Chapter 5 we have shown that the GUE two point correlation function can be obtained from the replica limit of the Toda lattice equation. In this section we will show that a similar integrable structure exists for the partially quenched QCD partition function. In particular, partition functions with bosonic quarks, fermionic quarks and the supersymmetric partition function form a single integrable hierarchy related by the Toda lattice equation [61, 102, 103].

13.1. Partition Function for N_f Fermionic Flavors

We consider the zero momentum limit of the QCD partition function with n fermionic flavors with mass m. The effective partition function for this case was already derived in section 10.1 and is given by

$$Z_\nu^{\text{eff}}(M) = \int_{U(n)} \det{}^\nu(U) e^{V\Sigma \text{Re} \text{Tr} MU}. \qquad (412)$$

In a diagonal representation of U this partition function can be rewritten as

$$\int \prod d\theta_k \prod_{k<l} |e^{i\theta_k} - e^{i\theta_l}|^2 e^{x\Sigma_k \cos \theta_k}, \qquad (413)$$

where $x = mV\Sigma$. By writing the Vandermonde determinant as

$$\prod_{k<l}(e^{i\theta_k} - e^{i\theta_l}) = \det[e^{ip\theta_q}]_{0\leq p\leq n-1, 1\leq q\leq n} \tag{414}$$

and expanding the determinant, the angular integrals can be written as modified Bessel functions. They can be recombined into a determinant as follows: [104, 105]

$$Z_n^\nu(x) = c_n \det[I_{\nu+k-l}(x)]_{0\leq k,l\leq n-1}. \tag{415}$$

Using recursion relations for Bessel functions this determinant can be rewritten as a $\tau-$function

$$Z_n^\nu(x) = \frac{c_n}{x^{n(n-1)}} \det[(x\partial_x)^{k+1} Z_1^\nu(x)]_{0\leq k,l\leq n-1}. \tag{416}$$

with

$$Z_1^\nu(x) = I_\nu(x). \tag{417}$$

The simplest way to prove this is to start from (416). To the columns of the matrix we apply the relation

$$x\partial_x[x^p I_{\nu+p}(x)] = (\nu + 2p)I_{\nu+p}(x) + x^{p+1}I_{\nu+p+1}(x) \tag{418}$$

starting with $p = 0$ and increasing p successively by 1 until all derivatives in the first row are gone. The first term in the recursion relation is canceled by the addition of a multiple of the previous column. We arrive at

$$Z_n = c_n \det[(x\partial_x)^k I_{\nu+l}(x)]_{0\leq k,l\leq n-1}. \tag{419}$$

Next we apply the recursion relation

$$x\partial_x[x^p I_{\nu+p}(x)] = x^{p+1}I_{\nu+p-1}(x) - \nu x^p I_{\nu+p}(x) \tag{420}$$

starting with the last row. The second term in the recursion cancels by the addition of a multiple of the previous row. We continue until the second row, and after that we start again in the same way with the last row and continue until the third row. After repeating this procedure until all derivatives are gone we end up with the representation (415) which completes the proof.

Exercise. Work out the proof for $n = 2$.

13.2. The Bosonic Partition Function

We already have seen in section 9.3 that the Goldstone manifold for n bosonic quarks is given by $Gl(n)/U(n)$. Using the same invariance arguments as before we obtain the low energy effective partition function

$$Z_{-n}^\nu = \int_{Q\in Gl(n)/U(n)} \det^\nu(Q) e^{\frac{1}{2}V\Sigma \mathrm{Tr} M(Q+Q^{-1})}. \tag{421}$$

In this case Q can be diagonalized as

$$Q = U \operatorname{diag}(e^{s_k}) U^{-1} \tag{422}$$

so that an eigenvalue representation of this partition function is given by

$$\int \prod_k ds_k \prod_k e^{\nu s_k} \prod_{k<l} (e^{s_k} - e^{s_l})(e^{-s_k} - e^{-s_l}) e^{x \sum_k \cosh s_k}. \tag{423}$$

The Vandermonde determinants can again be written as

$$\prod_{k<l} (e^{s_k} - e^{s_l}) = \det[e^{ps_q}]_{0 \leq p \leq n-1, 1 \leq q \leq n} \tag{424}$$

and a similar expression for $s_k \to -s_k$. By expanding the two determinants, the integrals combine into modified Bessel functions which can be combined into a determinant as follows [106]

$$Z_{-n}(x) = c_{-n} \det[K_{\nu+k+l}(x)]_{0 \leq k,l \leq n-1}. \tag{425}$$

Using recursion relations for Bessel functions this determinant can be rewritten as a $\tau-$function

$$Z_{-n}(x) = \frac{c_{-n}}{x^{n(n-1)}} \det[(x \partial_x)^{k+l} Z_{-1}(x)]_{0 \leq k,l \leq n-1}. \tag{426}$$

with

$$Z_{-1}(x) = K_\nu(x). \tag{427}$$

This simply follows from the observation that $(-1)^\nu K_\nu(x)$ and $I_\nu(x)$ satisfy the same recursion relations.

13.3. Replica Limit and Toda Lattice Equation

Instead of the supersymmetric generating function one could have used the replica trick to calculate the resolvent. The fermionic replica trick is defined by

$$G(z) = \lim_{n \to 0} \frac{1}{n} \log Z_n^\nu(z). \tag{428}$$

and the bosonic replica trick by

$$G(z) = \lim_{n \to 0} \frac{1}{-n} \log Z_{-n}^\nu(z). \tag{429}$$

If we take the replica limit of the fermionic or bosonic partition functions directly, we will obtain a result that differs from the supersymmetric calculation. We will see

below that these problems can be avoided if the take the replica limit of the Toda lattice equation.

We now consider bosonic and fermionic partition functions with all masses equal to z which only depend on the combination

$$x = z\Sigma V. \tag{430}$$

Next we use the Sylvester identity [64, 27] which is valid for the determinant of an arbitrary matrix A. It is given by

$$C_{ij}C_{pq} - C_{iq}C_{pj} = \det(A)C_{ij;pq}, \tag{431}$$

where the C_{ij} are cofactors of the matrix A and the $C_{ij;pq}$ are double cofactors. By applying this identity to the determinant in (416) for $i = j = n - 1$ and $p = q = n$, and writing the cofactors as derivatives of the partition function, we easily derive the Toda lattice equation [107, 108]

$$(x\partial_x)^2 \log Z_n^v(x) = 2nx^2 \frac{Z_{n+1}^v(x)Z_{n-1}^v(x)}{[Z_n^v(x)]^2}. \tag{432}$$

This equation has also been derived as a consistency condition for QCD partition functions [109].

Exercise. Show that the cofactors that enter in the derivation of (432) can be written as derivatives of the partition function.

In the derivation of (432) we have only used that the fermionic partition function is a τ-function. Since the bosonic partition function is also a τ-function (see (426), it satisfies the same Toda lattice equation.

The two semi-infinite hierarchies are connected by

$$\lim_{n\to 0} \frac{1}{n}(x\partial_x)^2 \log Z_n^v(x). \tag{433}$$

By extending to Toda lattice hierarchy to include an additional spectator boson, it can be shown that [103]

$$\begin{aligned} \lim_{n\to 0} \frac{1}{n}(x\partial_x)^2 \log Z_n^v(x) &= \lim_{y\to x} x\partial_x(x\partial_x + y\partial_y) \log Z_{1,-1}(x|y) \\ &= x\partial_x \lim_{y\to x} x\partial_x \log Z_{1,-1}(x,y) \\ &= x\partial_x x G(x). \end{aligned} \tag{434}$$

For the resolvent we obtain the identity

$$x\partial_x x G(x) = 2x^2 Z_1^v(x) Z_{-1}^v(x), \tag{435}$$

which explains this factorization property of the resolvent. In the same way it can be shown the factorization of the susceptibility in a bosonic and a fermionic partition function [102].

Inserting the expressions for Z_1 and Z_{-1} we find [61]

$$G(x) = x(K_\nu(x)I_\nu(x) + K_{\nu-1}(x)I_{\nu+1}(x)) + a\frac{1}{x}, \tag{436}$$

where the integration constant a is fixed by the condition that in the sector of topological charge ν we have ν zero eigenvalues resulting in a contribution of ν/x to the resolvent. In the calculation with the supersymmetric method we found (436) without the topological term because we started from a generating function for the nonzero eigenvalues only. This result was first obtained from chiral Random Matrix Theory and has also been derived from the replica limit of a Painlevé equation [62].

14. DIRAC SPECTRUM AT NONZERO CHEMICAL POTENTIAL

In this Chapter we study the quenched microscopic spectrum of the QCD Dirac operator at nonzero chemical potential. Using the replica limit of the Toda lattice equation we obtain the exact analytical result [102].

14.1. General Remarks

At nonzero baryon chemical potential the Dirac operator is modified according to

$$D \to D + \mu\gamma_0. \tag{437}$$

This Dirac operator does not have any hermiticity properties and its eigenvalues are scattered in the complex plane [110, 111, 112, 113, 114]. For small μ we expect that the width of the cloud of eigenvalues $\sim \mu^2$ [110]. The average spectral density is given by

$$\rho(\lambda) = \langle \sum_k \delta^2(\lambda - \lambda_k)\rangle, \tag{438}$$

and the average resolvent is defined as usual by

$$G(z) = \frac{1}{V}\left\langle \sum_k \frac{1}{i\lambda_k + z}\right\rangle. \tag{439}$$

Using that $\partial_{z^*}1/z = \pi\delta^2(z)$ we easily derive

$$\partial_{z^*}G(z)|_{z=\lambda} = \frac{\pi}{V}\rho(\lambda). \tag{440}$$

In this case it is possible to write down a supersymmetric generating function for the resolvent. However, we did not succeed in evaluating this partition function and leave it as a challenge to the reader to solve the problem by this method. We will calculate the resolvent and the spectral density from the replica limit of the Toda lattice equation [102].

The replica limit of the spectral density is given by [115, 116, 117]

$$\rho(z,z^*) = \lim_{n\to 0}\frac{1}{\pi n}\partial_z\partial_{z^*}\log Z_n(z,z^*), \tag{441}$$

with generating function given by

$$Z_n(z,z^*) = \langle \det^n(D+\mu\gamma_0+z)\det^n(-D+\mu\gamma_0+z^*)\rangle. \tag{442}$$

The low energy limit of this generating function is a chiral Lagrangian that is determined by its global symmetries and transformation properties. By writing the product of the two determinants as [118]

$$\det(D+\mu\gamma_0+z)\det(-D+\mu\gamma_0+z^*) = \det\begin{pmatrix} id+\mu & 0 & z & 0 \\ 0 & id-\mu & 0 & z^* \\ z & 0 & id^\dagger+\mu & 0 \\ 0 & z^* & 0 & id^\dagger-\mu \end{pmatrix}, \tag{443}$$

where we have used the decomposition of the Dirac operator given in (291), we observe that the $U(2)\times U(2)$ flavor symmetry is broken by the chemical potential term and the mass term. Invariance is recovered by transforming the mass term as in the case of zero chemical potential and the chemical potential term by a local gauge transformation.

If we expand the chiral Lagrangian for the spectrum of quenched QCD at nonzero chemical potential we obtain the following types of terms

$$p^2\pi_a^2, \quad \mu p_0\pi_a^2, \quad \mu^2\pi_a^2, \quad m\Sigma\pi_a^2. \tag{444}$$

The zero momentum modes factorize from the partition function if

$$\mu \ll \frac{1}{L}, \quad \text{and} \quad \frac{z\Sigma}{F^2} \ll \frac{1}{L^2}. \tag{445}$$

In this domain the partition function is uniquely determinant by invariance arguments as in the case of zero chemical potential. In the addition to the mass term we obtain a term from the zero momentum part of the covariant derivative resulting in the partition function

$$Z_n(z,z^*) = \int_{U(2n)} dU\, e^{-\frac{F^2\mu^2 V}{4}\mathrm{Tr}[U,B][U^{-1},B]+\frac{\Sigma V}{2}\mathrm{Tr}M(U+U^{-1})}, \tag{446}$$

where

$$B = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & -\mathbf{1}_n \end{pmatrix}, \quad M = \begin{pmatrix} z\mathbf{1}_n & 0 \\ 0 & z^*\mathbf{1}_n \end{pmatrix}. \tag{447}$$

14.2. Integration Formula

By decomposing a $U(2n)$ matrix as

$$U = \begin{pmatrix} u_1 & \\ & u_2 \end{pmatrix} \begin{pmatrix} v_1 & \\ & v_2 \end{pmatrix} \begin{pmatrix} \sqrt{1-b^2} & b \\ b & -\sqrt{1-b^2} \end{pmatrix} \begin{pmatrix} v_1^\dagger & \\ & v_2^\dagger \end{pmatrix}, \qquad (448)$$

with $u_1, u_2, v_1 \in U(n)$, $v_2 \in U(n)/U^n(1)$ and b a diagonal matrix, the following integration formula can be proved [102]

$$\int_{U(2n)} dU \det{}^\nu U e^{\frac{1}{2} \text{Tr}[M(U+U^{-1})] + \sum_p a_p \text{Tr}[(UBU^{-1}B)^p]}$$

$$= \frac{c_n}{(xy)^{n(n-1)}} \det[(x\partial_x)^k (y\partial_y)^l Z_1^\nu(x,y)]_{0 \le k,l \le n-1}, \qquad (449)$$

where

$$Z_1^\nu(x,y) = \int_0^1 \lambda d\lambda I_\nu(\lambda x) I_\nu(-\lambda y) e^{2\sum_p a_p \cos(2p\cos^{-1}\lambda)}, \qquad (450)$$

and c_n is an n-dependent constant.

Exercise. Show that the Jacobian of of the transformation (448) from $U^{-1}dU$ to $u_1^{-1}du_1, u_2^{-1}du_2, v_1^{-1}dv_1, v_2^{-1}dv_2$ and $\prod_k d\lambda_k$ is given by

$$J = \prod_{1 \le k < l \le n} (\lambda_k^2 - \lambda_l^2)^2 \prod_{k=1}^n (2\lambda_k), \qquad (451)$$

where $\lambda_k = \sqrt{1 - b_k^2}$.

Exercise. Show that for $n = 1$ and all $a_p = 0$ the integral $Z_1^\nu(x,y)$ is given by

$$Z_1^\nu(x,y)|_{a_p=0} = \int_0^1 \lambda d\lambda I_\nu(\lambda x) I_\nu(-\lambda y)$$

$$= \frac{y I_\nu(x) I_{\nu-1}(-y) + x I_{\nu-1}(x) I_\nu(-y)}{x^2 - y^2}. \qquad (452)$$

It can be interpreted as the QCD partition function with two different masses [119].

14.3. Toda Lattice Equation

Using the integration formula (449) for $p = 1$ we find that the zero momentum partition function $Z_n^\nu(z, z^*)$ (see eq. (446)) is given by

$$Z_n^\nu(z, z^*) = \frac{c_n}{(zz^*)^{n(n-1)}} \det[(z\partial_z)^k (z^*\partial_{z^*})^l Z_1^\nu(z, z^*)]_{0 \le k,l \le n-1}, \qquad (453)$$

where

$$Z_1^\nu(z,z^*) = \int_0^1 \lambda d\lambda\, e^{-2VF^2\mu^2(\lambda^2-1)} |I_\nu(\lambda z V\Sigma)|^2. \tag{454}$$

By applying the Sylvester identity to the determinant in (453) for $i = j = n-1$ and $p = q = n$ and expressing the cofactors as derivatives, we find a recursion relation that can be written in the form of a Toda lattice equation

$$z\partial_z z^* \partial_{z^*} \log Z_n^\nu(z,z^*) = \frac{\pi n}{2}(zz^*)^2 \frac{Z_{n+1}^\nu(z,z^*)Z_{n-1}^\nu(z,z^*)}{[Z_n^\nu(z,z^*)]^2}. \tag{455}$$

For the spectral density we find the simple expression ($Z_0^\nu(z,z^*) = 1$)

$$\rho(z,z^*) = \lim_{n\to 0} \frac{1}{\pi n} \partial_z \partial_{z^*} \log Z_n^\nu(z,z^*) = \frac{zz^*}{2} Z_1^\nu(z,z^*)Z_{-1}^\nu(z,z^*). \tag{456}$$

What remains to be done is to calculate the partition function with one bosonic and one conjugate bosonic quark which will be completed in the next subsection.

14.4. The Bosonic Partition Function

In this subsection we evaluate the low energy limit of the QCD partition function at nonzero chemical potential for one bosonic quark and one conjugate bosonic quark. Because of convergence requirements, the inverse determinants of nonhermitian operators have to regularized [120, 121]. This is achieved by expressing them as the determinant of a larger Hermitian operator

$$\det{}^{-1}\begin{pmatrix} z & id+\mu \\ id^\dagger+\mu & z \end{pmatrix} \det{}^{-1}\begin{pmatrix} z^* & -id+\mu \\ -id^\dagger+\mu & z^* \end{pmatrix} \tag{457}$$

$$= \lim_{\varepsilon\to 0} \det{}^{-1}\begin{pmatrix} \varepsilon & 0 & z & id+\mu \\ 0 & \varepsilon & id^\dagger+\mu & z \\ z^* & -id+\mu & \varepsilon & 0 \\ -id^\dagger+\mu & z^* & 0 & \varepsilon \end{pmatrix}.$$

To better expose the symmetries of this problem we rewrite the inverse determinant in the r.h.s. of this equation as

$$\int d\phi_k d\phi_k^* \exp[i \begin{pmatrix} \phi_1^* \\ \phi_2^* \\ \phi_3^* \\ \phi_4^* \end{pmatrix}^T \begin{pmatrix} \varepsilon & z & id+\mu & 0 \\ z^* & \varepsilon & 0 & id-\mu \\ -id^\dagger+\mu & 0 & \varepsilon & -z^* \\ 0 & -id^\dagger-\mu & -z & \varepsilon \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}]. \tag{458}$$

In this case the mass matrices are given by

$$\zeta_1 = \begin{pmatrix} \varepsilon & z \\ z^* & \varepsilon \end{pmatrix} \qquad \text{and} \qquad \zeta_2 = \begin{pmatrix} \varepsilon & -z^* \\ -z & \varepsilon \end{pmatrix}. \tag{459}$$

The two mass matrices are related by

$$\zeta_2 = -I\zeta_1 I \qquad \text{with} \qquad I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{460}$$

For $\mu = 0$ and $\zeta_1 = \zeta_2 = 0$ the basic symmetry[2] of the partition function is $Gl(2) \times Gl(2)$,

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rightarrow U_1 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \qquad \begin{pmatrix} \phi_1^* \\ \phi_2^* \end{pmatrix} \rightarrow \begin{pmatrix} \phi_1^* \\ \phi_2^* \end{pmatrix} U_2^{-1}$$

$$\begin{pmatrix} \phi_3 \\ \phi_4 \end{pmatrix} \rightarrow U_2 \begin{pmatrix} \phi_3 \\ \phi_4 \end{pmatrix} \qquad \begin{pmatrix} \phi_3^* \\ \phi_4^* \end{pmatrix} \rightarrow \begin{pmatrix} \phi_3^* \\ \phi_4^* \end{pmatrix} U_1^{-1}. \tag{461}$$

This symmetry can be extended to nonzero mass or chemical potential if we adopt the transformation rules

$$\begin{aligned} \zeta_1 &\rightarrow U_2\zeta_1 U_1^{-1}, & \zeta_2 &\rightarrow U_1\zeta_2 U_2^{-1}, \\ \mu_1 &\rightarrow U_2\mu_1 U_2^{-1}, & \mu_2 &\rightarrow U_1\mu_2 U_1^{-1}, \end{aligned} \tag{462}$$

where μ_1 is the chemical potential matrix that is added id and μ_2 is the chemical potential matrix that is added to $-id^\dagger$. These matrices are introduced for the sake of discussing the transformation properties of the partition function (458) and will ultimately be replaced by their original values

$$\mu_1 = \mu_2 = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}. \tag{463}$$

The chiral symmetry is broken spontaneously by the chiral condensate to $Gl(2)$. Because of convergence of the bosonic integral, the Goldstone manifold is not $Gl(2)$ but rather $Gl(2)/U(2)$, i.e. the coset of positive definite matrices as in the case of zero chemical potential. Under a $Gl(2) \times Gl(2)$ transformation the Goldstone fields transform as

$$Q \rightarrow U_2 Q U_1^{-1}. \tag{464}$$

The low energy effective partition function should have the same transformation properties as the microscopic partition function (458). To second order in μ and first order in the mass matrix we can write down the following invariants

$$\text{Tr}\zeta_1 Q, \quad \text{Tr}\zeta_2 Q^{-1}, \quad \text{Tr}Q\mu_2 Q^{-1}\mu_1, \quad \text{Tr}\mu_1\mu_2. \tag{465}$$

We also have the discrete symmetry that the partition function is invariant under the interchange of ζ_1 and ζ_2. This symmetry implies that the coefficients of the two mass terms in the effective partition function are the same. Using that the integration measure

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[2] This is the symmetry when we disregard convergence. Taking convergence of the bosonic integrals into account the vector symmetry is $U(2)$ and the axial symmetry is $Gl(2)/U(2)$.

on positive definite Hermitian matrices is given by $dQ/\det^2 Q$, we finally arrive at the effective partition function[3]

$$Z^{\nu}_{-1}(z,z*) = \int_{Q\in Gl(2)/U(2)} \frac{dQ}{\det^2(Q)}\det^{\nu}(Q)e^{-\frac{F^2\mu^2 V}{4}\mathrm{Tr}[Q,B][Q^{-1},B]+\frac{i\Sigma V}{2}\mathrm{Tr}(\zeta_1 Q+\zeta_2 Q^{-1})}.$$

(466)

Exercise. Prove that for a parameterization of positive definite 2×2 matrices as $Q = U\mathrm{diag}(e^{s_1},e^{s_2})U^{-1}$ the invariant measure is given by

$$\frac{dQ}{\det^2 Q} = (e^{s_1}-e^{s_2})(e^{-s_1}-e^{-s_2})ds_1 ds_2 dU,$$

(467)

where dU is the invariant measure on $U(2)/U(1)\times U(1)$.

To evaluate the integral (466) we do not use the parameterization of the above exercise but rather

$$Q = e^t \begin{pmatrix} e^r\cosh s & e^{i\theta}\sinh s \\ e^{-i\theta}\sinh s & e^{-r}\cosh s \end{pmatrix}.$$

(468)

where

$$r\in\langle-\infty,\infty\rangle,\quad s\in\langle-\infty,\infty\rangle,\quad t\in\langle-\infty,\infty\rangle,\quad \theta\in\langle 0,\pi\rangle\quad.$$

(469)

Using that the Jacobian from the measure $dQ/\det^2 Q$ to the measure $drdsdtd\theta$ is given by

$$J = 4e^{4t}\cosh s\sinh s,$$

(470)

we find the partition function ($z = x+iy$)

$$Z^{\nu}_{-1}(z,z^*) = \lim_{\varepsilon\to 0} C_\varepsilon \int drdsdtd\theta\cosh s|\sinh s|e^{2\nu t} \tag{471}$$
$$\times e^{i\frac{V\Sigma}{2}(4x\sinh s\cosh t\cos\theta-4y\sinh s\sinh t\sin\theta+4\varepsilon\cosh r\cosh s\cosh t)-\mu^2 F^2 V(1+2\sinh^2 s)}.$$

The integral over r is equal to $2K_0(2N\varepsilon\cosh s\cosh t)$ with leading singularity given by $\sim -\log\varepsilon$. This factor is absorbed in the normalization of the partition function. Then the integral over θ gives a Bessel function. Introducing $u = \sinh s$ as new integration variable we obtain [102]

$$Z^{\nu}_{-1}(z,z^*) = C_{-1}\int_{-\infty}^{\infty}dt\int_0^{\infty}due^{2\nu t}J_0(2V\Sigma u(x^2\cosh^2 t+y^2\sinh^2 t)^{1/2})e^{-\mu^2 F^2 V(1+2u^2)}.$$

(472)

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[3] Notice that the Q variables used in this lecture are the transpose the Q variables used in [102]. This simplifies the discussion of the transformation properties

To perform the integral over u we use the known integral

$$\int_0^\infty dx\, x^{a+1} e^{-\alpha x^2} J_a(\beta x) = \frac{\beta^a}{(2\alpha)^{a+1}} e^{-\beta^2/4\alpha}. \tag{473}$$

This results in

$$Z^\nu_{-1}(z,z^*) = \frac{C_{-1}e^{-V\mu^2 F^2}}{4\mu^2 F^2 V} \int_{-\infty}^\infty dt\, e^{2\nu t} e^{-\frac{V(x^2\cosh^2 t + y^2\sinh^2 t)}{2\mu^2 F^2}}. \tag{474}$$

Using that $\cosh^2 t = \frac{1}{2} + \frac{1}{2}\cosh 2t$ and $\sinh^2 t = -\frac{1}{2} + \frac{1}{2}\cosh 2t$, the integral over t can be rewritten as a modified Bessel function resulting in [102]

$$Z^\nu_{-1}(z = x+iy, z^*) = C_{-1}\, e^{\frac{V\Sigma^2(y^2-x^2)}{4\mu^2 F^2}} K_\nu\left(\frac{V\Sigma^2(x^2+y^2)}{4\mu^2 F^2}\right). \tag{475}$$

14.5. The Dirac Spectrum at Nonzero Chemical Potential

The final result for the quenched spectral density is obtained by substituting the partition functions $Z^\nu_1(z,z^*)$ and $Z^\nu_{-1}(z = x+iy, z^*)$ in expression (456) obtained from the replica limit of the Toda lattice equation. We find,

$$\begin{aligned}
\rho(x,y) &= \frac{V^3\Sigma^4}{2\pi F^2\mu^2}(x^2+y^2) e^{\frac{V\Sigma^2(y^2-x^2)}{4\mu^2 F^2}} K_\nu\left(\frac{V\Sigma^2(x^2+y^2)}{4\mu^2 F^2}\right) \\
&\quad \times \int_0^1 \lambda d\lambda\, e^{-2VF^2\mu^2\lambda^2} |I_\nu(\lambda z V\Sigma)|^2.
\end{aligned} \tag{476}$$

The normalization constant has been chosen such that the $\mu \to 0$ limit of $\rho(x,y)$ for large y is given by $\Sigma V/\pi$ (see below).

In Fig. 4 we show a graph of the spectral density for $y = 0$ and $\mu^2 F^2 V = 16$ (solid curve) plotted as the dimensionless ratio

$$\rho_s(x,y) = \frac{\rho(x,y)}{\Sigma^2 V^2}. \tag{477}$$

Also shown is the result where the Bessel function K_ν is replaced by its asymptotic expansion (dotted curve). This result has been obtained from a nonhermitian eigenvalue model [122] that is not in the universality class of QCD. An important difference between the two results is that the spectral density (476) is quadratic in x for $x \to 0$, whereas the result given by the dotted curve is linear in x for $x \to 0$.

In the limit $\mathrm{Re}(z)\Sigma/\mu^2 F^2 \ll 1$ the upper limit of the integral in (476) can be extended to infinity. Using that

$$\int_0^\infty \lambda d\lambda\, e^{-2VF^2\mu^2\lambda^2} |I_\nu(\lambda z V\Sigma)|^2 = \frac{1}{4\mu^2 F^2 V^2} e^{\frac{(z^2+z^{*2})\Sigma^2 V}{8\mu^2 F^2}} I_\nu\left(\frac{zz^* V\Sigma^2}{4\mu^2 F^2}\right) \tag{478}$$
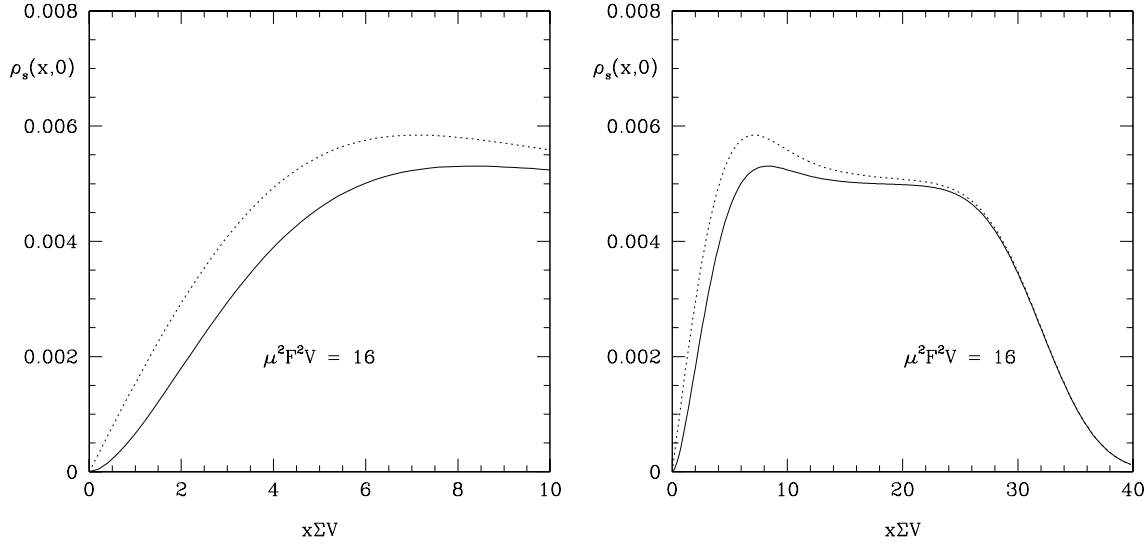
FIGURE 4. The quenched spectral density at nonzero chemical potential in the ergodic domain of QCD (full curve). The dotted curve represents the result obtained from a nonhermitian eigenvalue model. (Taken from [84].)

the spectral density can be expressed as

$$\rho(x,y) = \frac{2}{\pi}u^2 zz^* K_\nu(zz^* u) I_\nu(zz^* u) \qquad \text{with} \qquad u = \frac{V\Sigma^2}{4\mu^2 F^2}. \tag{479}$$

Therefore, the spectral density becomes a universal function that only depends on a single parameter u. This parameter can be rewritten in a more physical way as

$$u = \pi \rho_{\text{asym}}(x,y). \tag{480}$$

For the dimensionless ratio we obtain [4]

$$\frac{\rho(x,y)}{\rho_{\text{asym}}(x,y)} = 2uzz^* K_\nu(zz^* u) I_\nu(zz^* u) \tag{481}$$

which is universal combination that depends only on $zz^* u$.

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[4] This formula arose as a result of a discussion with Tilo Wettig.

In the thermodynamic limit the Bessel functions can be approximated by their asymptotic limit,

$$K_v(z) = \sqrt{\frac{\pi}{2z}} e^{-z}, \quad I_v(z) = \frac{1}{\sqrt{2\pi z}}(e^z + ie^{-z}), \quad I_v(z^*) = \frac{1}{\sqrt{2\pi z^*}}(e^{z^*} - ie^{-z^*}). \quad (482)$$

This results in

$$\rho(x,y) = \frac{V^2\Sigma^2}{2\pi F\mu\sqrt{2\pi V}} \int_0^1 d\lambda\, e^{-2VF^2\mu^2(\lambda - \frac{|x|\Sigma}{2F^2\mu^2})^2}. \quad (483)$$

For $V \to \infty$ the integral over λ can be performed by a saddle point approximation. If the saddle point is outside the range $[0,1]$ we find zero for $V \to \infty$. This results in the spectral density

$$\rho(x,y) = \frac{V\Sigma^2}{4\pi\mu^2 F^2} \quad \text{for} \quad |x| < \frac{2F^2\mu^2}{\Sigma}. \quad (484)$$

and $\rho(x,y) = 0$ outside this strip. This result is in agreement with the mean field analysis of the effective partition function [118]. For the integrated eigenvalue density we find

$$\int_{-\infty}^{\infty} dx\rho(x,u) = \frac{\Sigma V}{\pi} \quad (485)$$

in agreement with the eigenvalue density at $\mu = 0$.

15. CONCLUSIONS

The supersymmetric method is a powerful method to analyze random matrix ensembles. This approach makes it possible to formulate the problem in terms of symmetries and the spontaneous breaking of these symmetries. The final result for universal correlation functions is uniquely given by a partition function of Goldstone modes interacting according to an effective Lagrangian with the symmetries of the microscopic partition function. Convergence requirements necessary lead to a Goldstone manifold that is the product of a compact and a noncompact submanifold. In the final answer for the correlation function this structure is present in the form of compact and noncompact integrals and is therefore the natural formulation of the problem.

In these lectures we have given an elementary introduction to Random Matrix Theory and the supersymmetric method. We have applied this method to four problems which are technically the least demanding: the one-point function of the Gaussian Unitary Ensemble (GUE), the two-point function of the Gaussian Unitary Ensemble and the one-point function of the chiral Gaussian Unitary Ensemble both as zero and at nonzero chemical potential. The one-point function of the GUE is not a universal quantity and was only discussed to illustrate the method. The two-point function of the GUE and the one-point function of the chGUE are universal quantities. They are relevant for eigenvalue correlations of Hamiltonian systems with broken time reversal invariance

and no other symmetries and for the distribution of the small eigenvalues of the QCD Dirac operator, respectively.

We have also shown that for $\beta = 2$ the supersymmetric partition function connects the semi-infinite bosonic and the semi-infinite fermionic Toda lattice hierarchy of partition functions. This makes it possible to obtain the second derivative of the partition function from the replica limit of the Toda lattice equation. The final result is the product of a bosonic (noncompact) and a fermionic (compact) integral. In this approach there is no need to analyze a supersymmetric non-linear σ-model. The factorized structure of the final answer is a direct consequence of the Toda lattice equation and make this the minimal approach for the evaluation a supersymmetric partition function. It is still an open problem whether integrable structures play a similar role for other values of the Dyson index.

The supersymmetric method is not the only method to derive nonperturbative results for Random Matrix Theories. Perhaps the best known method is the orthogonal polynomial method. This method starts from the joint eigenvalue distribution and exploits orthogonality relations to perform the integrals that lead to the correlation functions we are interested in. Therefore, this method fails for problems that can not be formulated in terms of a joint eigenvalue density. Most notably, the statistical theory of S-matrix fluctuations and the theory of parametric correlations have only been solved by means of the supersymmetric method.

A disadvantage of the supersymmetric method is that it requires a thorough knowledge of supersymmetry and superanalysis. A straightforward evaluation of the Grassmann integrals leads to a factorial proliferation of terms and does not work in all but the simplest cases. To find the right variables to solve the supersymmetric integrals is often an arduous task and one wonders why the symmetries of the problem do not provide us with a more directed path to the final result. One hint that more progress can be made is that in some cases where the supersymmetric method fails because of technical problems the problem can be solved exactly by means of the replica limit of the Toda lattice equation. We have illustrated this for the quenched spectral density of the QCD Dirac operator at nonzero chemical potential. Meanwhile, this problem has also been solved by means of the orthogonal polynomial method [123, 124], but the derivation of this solution from a supersymmetric nonlinear σ-model has remained elusive.

ACKNOWLEDGMENTS

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